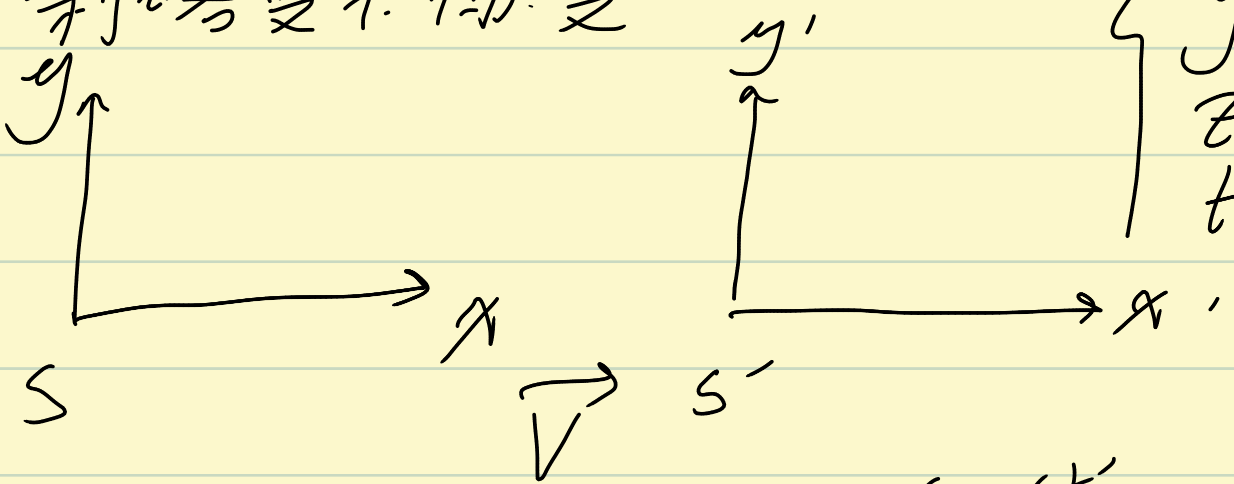
电动力学

一. 从Maxwell方程组到狭义相对论

1. Lorentz变换的由来

$$
\begin{cases}
\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \\
\vec{\nabla}\cdot\vec{H} = 0 \\
\vec{\nabla}\cdot\vec{E} = 4\pi\rho \\
\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\left(\vec{j} + \frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}\right)
\end{cases}
$$

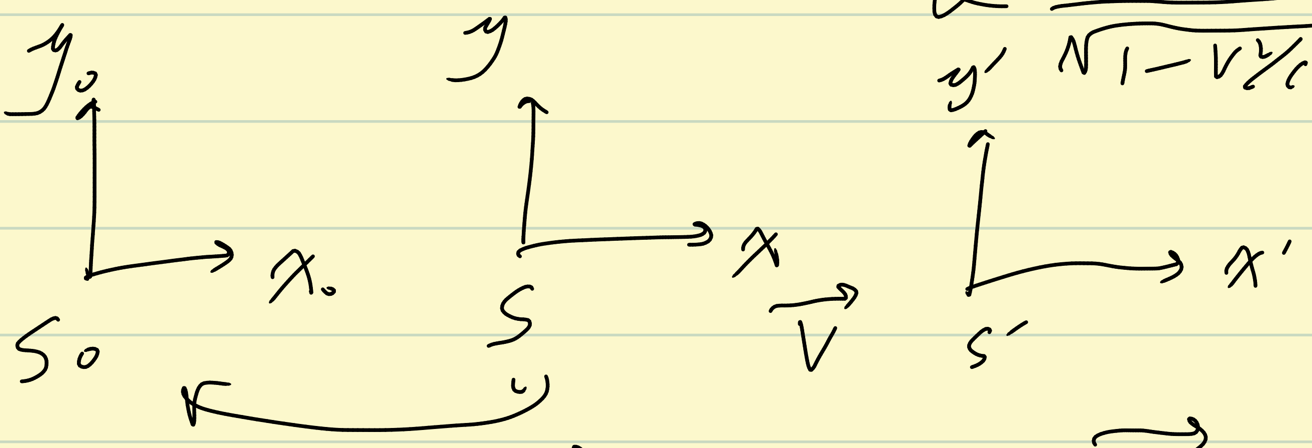
在伽利略变不协变



$$
\begin{cases}
x = x' + Vt \\
y = y' \\
z = z' \\
t = t'
\end{cases}
$$

Lorentz 数学trick

$$
\begin{cases}
x = \alpha \dfrac{x' + Vt'}{\sqrt{1 - V^2/c^2}} \\
y = \alpha y' \\
z = \alpha z' \\
t = \alpha \dfrac{t' + (\frac{V}{c^2})x}{\sqrt{1 - V^2/c^2}}
\end{cases}
$$

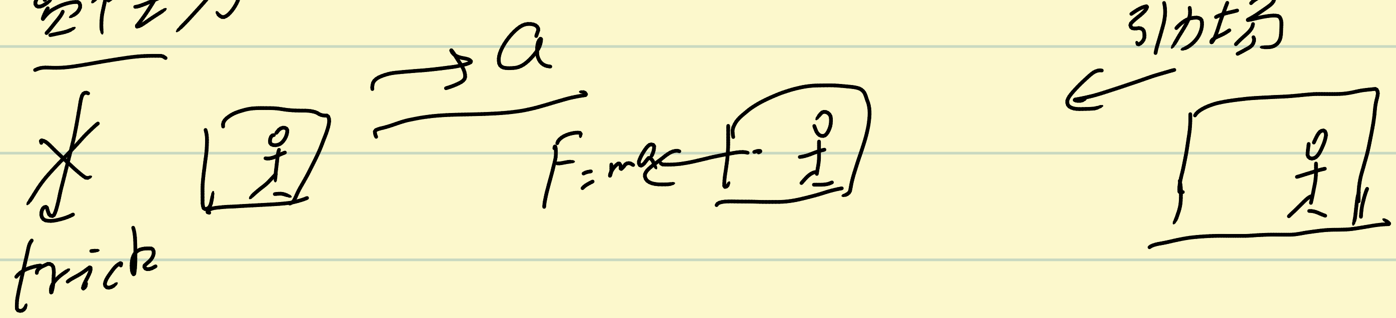


$\vec{v_1} + \vec{V} \neq \vec{v_2}$

$v_{SS_0} = \vec{v_1}$ $\quad$ $v_{S'S_0} = \vec{v_2}$

长度和时间间隔 $\sqrt{1 - v^2/c^2}$

惯性力



"等效原理"

同理，Lorentz → 变换. 以太多余

→ 狭义相对论

2. 狭义相对论

①狭义相对性原理：O

②光速不变 → 实验

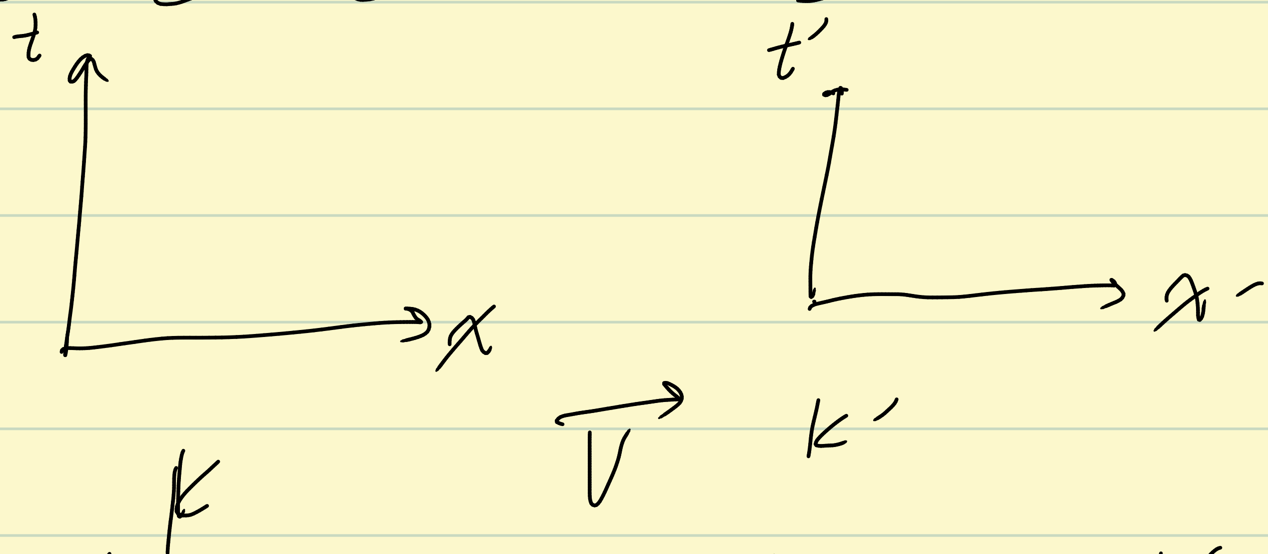
→ $\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$

→ 数学表述

四维时空 (t, x, y, z) 事件 → 世界点

粒子的运动 → 世界线

匀速直线运动世界线 → 直线



K'系中 有一个光信号 (t_1', x_1', y_1', z_1') (t_2', x_2', y_2', z_2')

$$s_{12}'^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 - c^2(t_2' - t_1')^2 = 0$$

K系中

$$s_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = 0$$

事件1 (t_1, x_1, y_1, z_1)　　事件2 (t_2, x_2, y_2, z_2)

定义事件1,2的间隔

$$s_{12}=\sqrt{c^2(t_2-t_1)^2-(x_2-x_1)^2-(y_2-y_1)^2-(z_2-z_1)^2}$$

$$ds^2=c^2dt-dx^2-dy^2-dz^2$$

$ds^2=0$　　$ds'^2=0$

$\boxed{ds^2=a\,ds'^2}$?　　K　$\vec{v}$　K'

$a(|\vec{v}|)$　　$v_{21}=\vec{2}$

K_0　K_1　K_2

$\vec{v}_{10}=\vec{v}_1$　$\vec{v}_{20}=\vec{v}_2$

$$ds_0^2=a(|\vec{v}_1|)ds_1^2=a(|\vec{v}_2|)ds_2^2$$

$$ds_1^2=a(|\vec{v}_{12}|)ds_2^2 \longrightarrow a(|\vec{v}_{12}|)=\frac{a(|\vec{v}_2|)}{a(|\vec{v}_1|)}$$

$\Rightarrow a=1$.

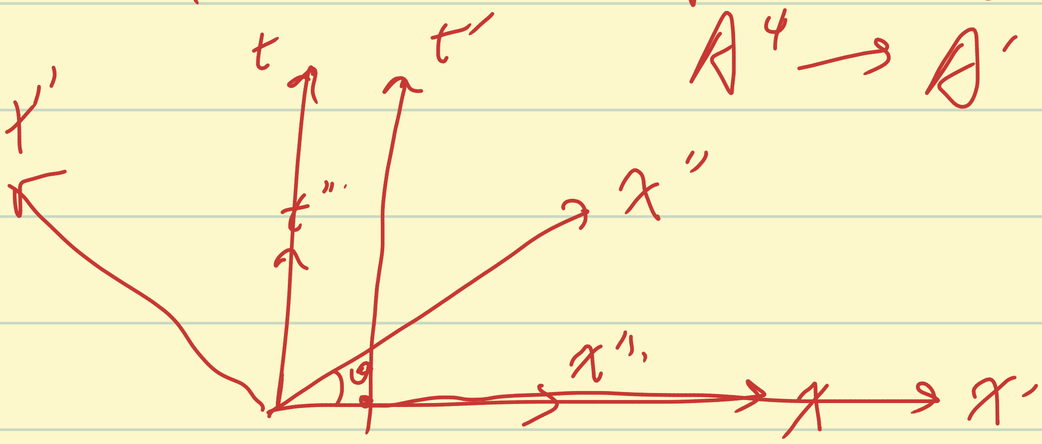
$ds^2=ds'^2 \longrightarrow s=s'$

结论：原理②前提下，间隔不变.

$ds^2=a\,ds^3$　$ds^2=a(e^{ds^2}-1)$,　$ds^2=a\ln(ds^2+1)$

伽利略变换 or 洛伦兹变换 $\longrightarrow$ 仿射变换

世界：四维仿射空间 A^4



$A^4 \longrightarrow A'^4$

①匀速直线运动

②线段 $\longrightarrow$ 直线

阿诺德《经典力学的数学方法》

K. $A(t_1, x_1, y_1, z_1)$ $B(t_2, x_2, y_2, z_2)$

令 $t_{12} = t_2 - t_1$. $l_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2$

K′ $s_{12}'^2 = c^2 t_{12}'^2 - l_{12}'^2$

① K′ A、B 同空间点 $\rightarrow l_{12}' = 0$

$\Rightarrow s_{12}^2 = c^2 t_{12}'^2 - l_{12}'^2 = c^2 t_{12}'^2 > 0$. s_{12} 类时间隔.

K′ $t_{12}' = \frac{s_{12}}{c}$.

A.B 在同一个物体上、 s_{12} 永远类时.

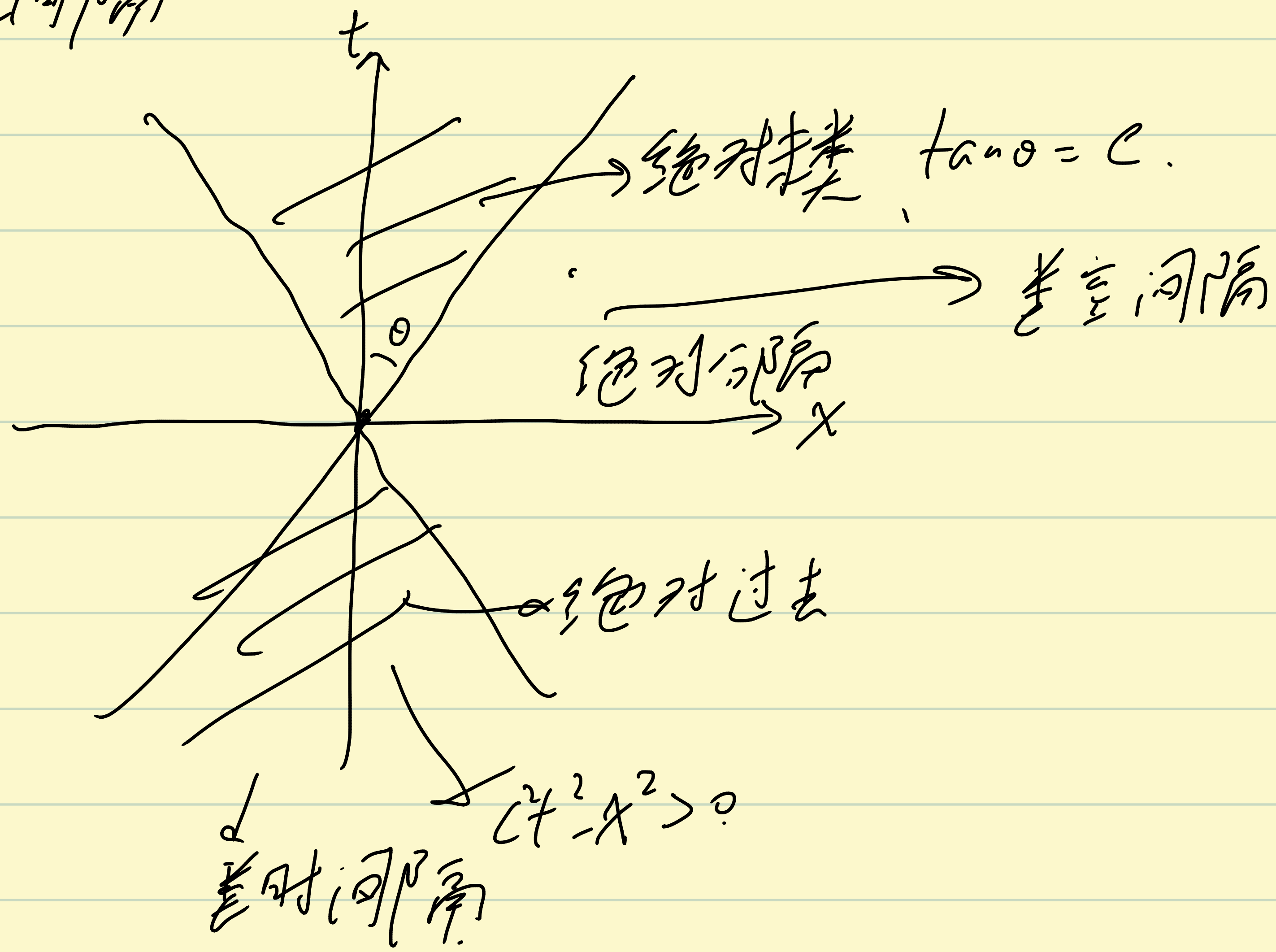
$l_{12} < c t_{12} \rightarrow v < c$.

② K′, A、B 同时间点 $\rightarrow t_{12}' = 0$.

$\Rightarrow s_{12}^2 = c^2 t_{12}^2 - l_{12}^2 = -l_{12}'^2 < 0$. s_{12} 虚数 类空间隔

$l_{12}' = i s_{12}$.

③ K′ 类光间隔

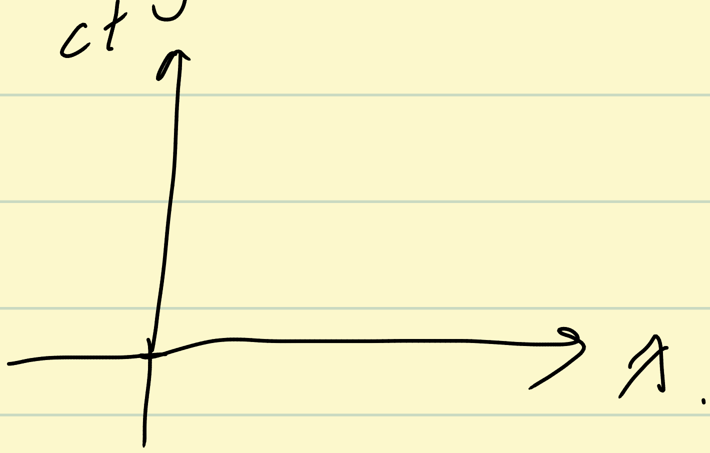


$s^2 = ct^2 - x^2 - y^2 - z^2$ 不变.

欧氏空间. 平移+旋转. $\vec{a}\cdot\vec{b} = xy_1 + x_2y_2 + x_3y_3$.

闵氏四维, $s^2 = c^2t^2 - x^2 - y^2 - z^2$

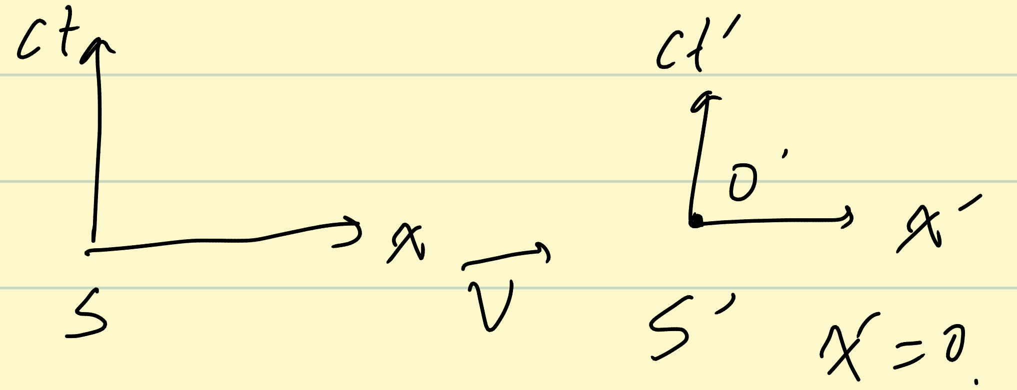
(ct, x, y, z)



$x^2 + y^2$ $\quad \sin^2\theta + \cos^2\theta = 1$

$ct^2 - x^2$ $\quad \cosh^2\theta - \sinh^2\theta = 1$

$$\begin{cases} x = x'\cosh\psi + ct'\sin\psi \\ ct = x'\sinh\psi + ct'\cos\psi \end{cases}$$



$x'=0$

$\frac{x}{t} = \vec{v}$

$$x = ct'\sinh\psi$$
$$ct = ct'\cosh\psi$$

$\Rightarrow \tanh\psi = \frac{x}{ct} = \frac{v}{c}$

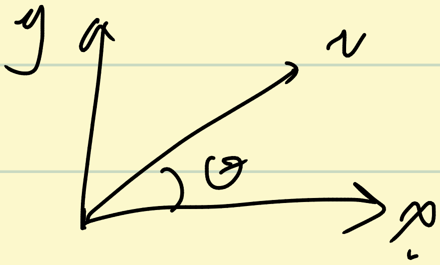
$\rightarrow \sinh\psi = \frac{v/c}{\sqrt{1-v^2/c^2}}, \quad \cosh\psi = \frac{1}{\sqrt{1-v^2/c^2}}$

$$\begin{cases} x = \dfrac{x'+vt'}{\sqrt{1-v^2/c^2}} \\ y = y' \\ z = z' \\ t = \dfrac{t'+\frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}} \end{cases}$$

速度变换与光行差公式

$$dx = \frac{dx'+v\,dt'}{\sqrt{1-v^2/c^2}} \quad dy = dy', \quad dz = dz', \quad dt = \frac{dt'+\frac{v}{c^2}dx'}{\sqrt{1-v^2/c^2}}$$

$$\begin{cases} u_x = \dfrac{u_x'+v}{1+\frac{u_x' v}{c^2}} \\ u_y = \dfrac{u_y'\sqrt{1-v^2/c^2}}{1+\frac{u_x' v}{c^2}} \\ u_z = \dfrac{u_z'\sqrt{1-v^2/c^2}}{1+\frac{u_x' v}{c^2}} \end{cases}$$



$$\begin{cases} v_x = v\cos\theta \\ v_y = v\sin\theta \end{cases} \qquad \begin{cases} v'_x = v'\cos\theta' \\ v'_y = v'\sin\theta' \end{cases}$$

$$\tan\theta = \frac{v_y}{v_x} = \frac{v'\sin\theta'\sqrt{1-V^2/c^2}}{v'\cos\theta' + V}.$$

$v = v' = c.$

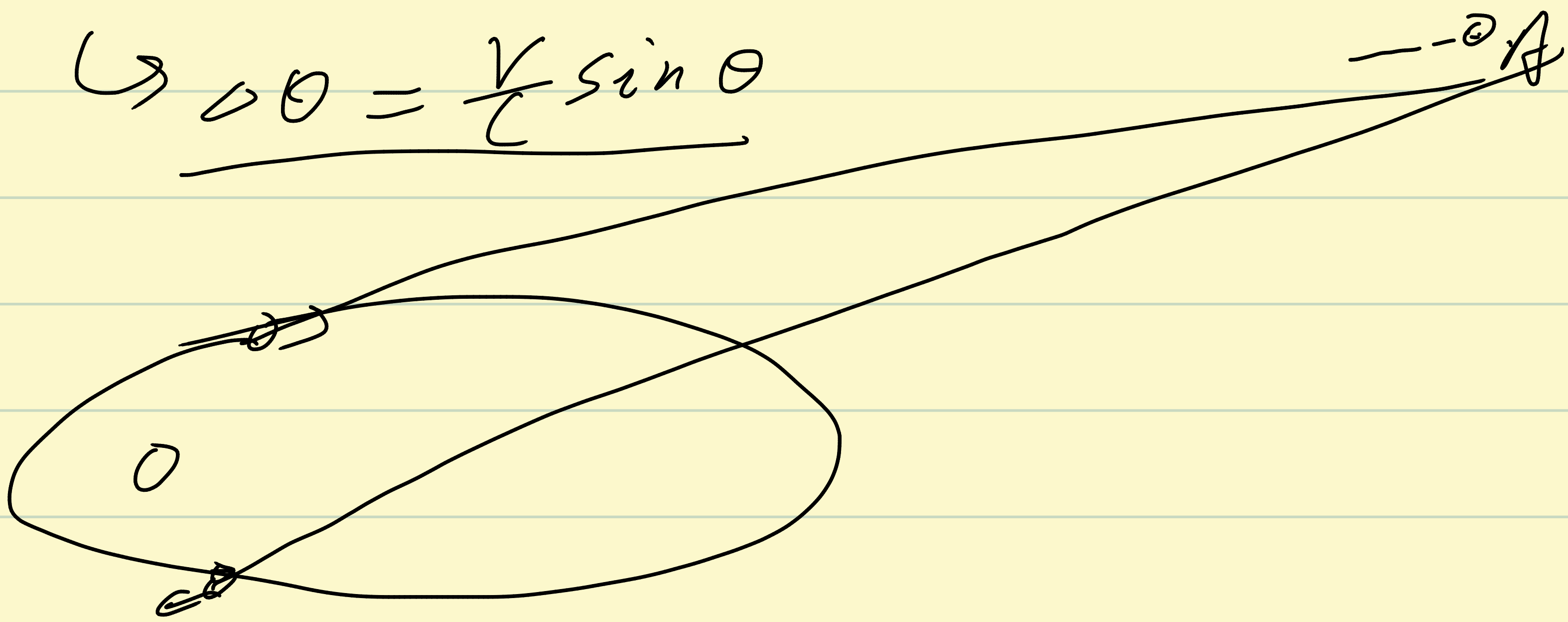
$$\tan\theta = \frac{\sqrt{1-V^2/c^2}}{\frac{V}{c} + \cos\theta'}\sin\theta'$$

$$\sin\theta = \frac{\sqrt{1-V^2/c^2}}{1+\frac{V}{c}\cos\theta'}\sin\theta' \qquad V \ll c.$$

$$\approx \left(1-\frac{V}{c}\right)\cos\theta'\sin\theta'$$

$$\rightarrow \quad \sin\theta - \sin\theta' \approx -\frac{V}{c}\cos\theta' \qquad \theta - \theta' = \Delta\theta.$$

$$\hookrightarrow \quad \underline{\Delta\theta = \frac{V}{c}\sin\theta}$$



二. 相对论力学

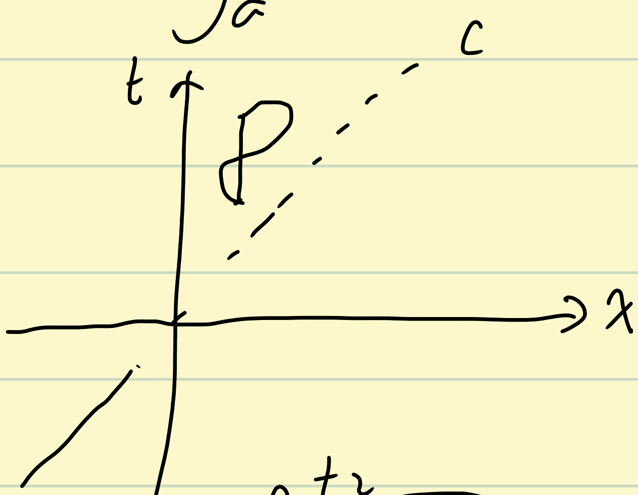
最小作用量原理: 力学体系 S 取小值, $\rightarrow \delta S = 0$.

S: 作用量

一个自由粒子 $\alpha\, ds$ $\qquad ds = c\,dt\sqrt{1-\frac{v^2}{c^2}}$

$$S = -\alpha\int_a^b ds, \quad \alpha > 0$$

$$c\int_0 \sqrt{1-\frac{v^2}{c^2}}\,dt$$



$$S = -\alpha c\int_{t_1}^{t_2}\sqrt{1-\frac{v^2}{c^2}}\,dt = \int_{t_1}^{t_2} L\,dt$$

$$L = -\alpha c\sqrt{1-\frac{v^2}{c^2}} \approx -\alpha c\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)$$

$c \to +\infty$

$$= \cancel{-\alpha c} + \frac{1}{2}\alpha\frac{v^2}{c}$$

$$= \frac{1}{2}\alpha\frac{v^2}{c} \iff L = \frac{1}{2}mv^2$$

$$\alpha = mc$$

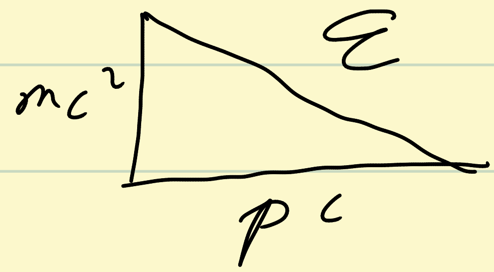
$$S = -mc\int_a^b ds, \quad L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{-mc^2\cdot\left(-\frac{2}{c^2}\vec{v}\right)}{2\sqrt{1-\frac{v^2}{c^2}}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \xrightarrow{c\to+\infty} m\vec{v}$$

$$\mathcal{E} = \vec{p}\cdot\vec{v} - L = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\xrightarrow{c\to+\infty} mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2$$

$$\mathcal{E}^2 = (pc)^2 + (mc^2)^2$$



$$\mathcal{H} = \sqrt{p^2c^2 + m^2c^4}$$
$$= c\sqrt{p^2 + m^2c^2}$$

$\frac{v}{c} \to 0$
$\Updownarrow$
$p \ll mc$
$$\approx mc^2\sqrt{\left(\frac{p}{mc}\right)^2 + 1}$$
$$= mc^2\left[1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2\right] = mc^2 + \frac{p^2}{2m}$$

$\vec{p} = \frac{\mathcal{E}\vec{v}}{c^2}$ $|\vec{v}| = c \longrightarrow |\vec{p}| = \frac{\mathcal{E}}{c}$

四维形式 $ds^2 = dx^i dx_i \to ds = \sqrt{dx^i dx_i}$

$$S = -mc\int_a^b ds$$

$$\delta S = -mc\int_a^b \delta ds = -mc\int_a^b \frac{dx^i d\delta x_i + d\delta x^i dx_i}{2\sqrt{dx^i dx_i}}$$
$$= -mc\int_a^b \frac{dx_i d\delta x^i}{ds}$$

$\longrightarrow$ 定义四维速度: $u^i = \frac{dx^i}{ds}$ $x^i = (ct, \vec{x})$

$$u^i = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\vec{v}}{c\sqrt{1-\frac{v^2}{c^2}}}\right)$$
$$x^i x_i = ds^2$$
$$u^i u_i = 1$$

$$\delta S = -mc\int_a^b \frac{dx_i d\delta x^i}{ds} = -mc\, u_i \delta x^i \Big|_a^b + \overset{=0}{mc\int_a^b \delta x^i \frac{du_i}{ds} ds}$$

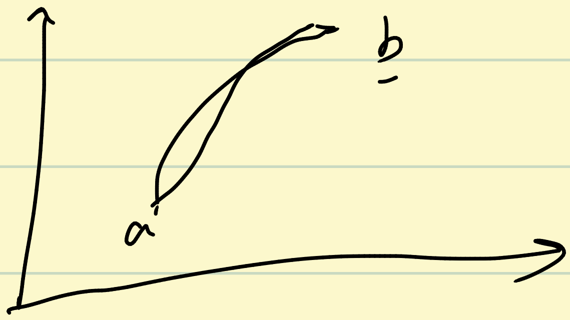
$$\delta x^i_{(a)} = \delta x^i_{(b)} = 0$$

$$\delta S = mc\int \frac{du_i}{ds}\,\delta x^i ds = 0$$

$\Downarrow$

0

自由粒子. $u_i \rightarrow$ 守恒.



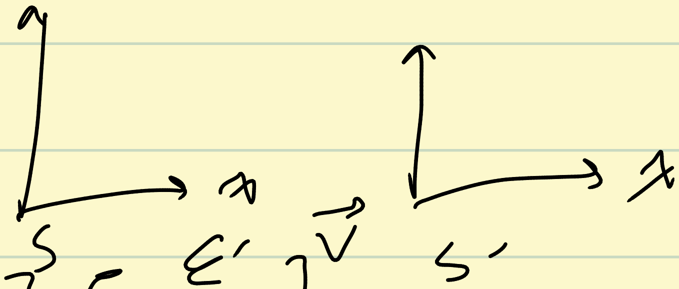
$$\delta S = -mc\, u_i\, \delta x^i \qquad \vec{p} = \frac{\partial S}{\partial q_i},\quad \mathcal{E} = -\frac{\partial S}{\partial t}.$$

定义四维动量矢量: $p_i = -\frac{\partial S}{\partial x^i}$

$$p_i = \left(\frac{\mathcal{E}}{c}, -\vec{p}\right) \qquad p^i = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)$$

$$p^i = mc\,u^i$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\quad \beta = \frac{v}{c}$$



$$\begin{bmatrix} \frac{\mathcal{E}}{c} \\ p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\mathcal{E}'}{c} \\ p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

$$\begin{cases} \mathcal{E} = \gamma\mathcal{E}' + \gamma\beta p'_x c = \dfrac{\mathcal{E}' + V p'_x}{\sqrt{1-\frac{V^2}{c^2}}} \\ p_x = \gamma\beta\dfrac{\mathcal{E}'}{c} + \gamma p'_x = \dfrac{p'_x + \frac{V}{c^2}\mathcal{E}'}{\sqrt{1-\frac{V^2}{c^2}}} \\ p_y = p'_y \\ p_z = p'_z \end{cases}$$

$$u^i u_i = 1$$

$$p^i p_i = m^2 c^2$$

$p_i = -\frac{\partial S}{\partial x^i}$, $p^i = -\frac{\partial S}{\partial x_i}$

$m^2c^2 = p_i p^i = \frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x_i} = g^{ik}\frac{\partial S}{\partial x^k}\frac{\partial S}{\partial x^i}$

Hamilton-Jacob方程.

$m^2c^2 = \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right]$

$\mathcal{E} = -\frac{\partial S}{\partial t}$

$S' = S + mc^2 t$

$\cancel{m^2c^2} = \frac{1}{c^2}\left(\frac{\partial S'}{\partial t}\right)^2 - 2m\frac{\partial S'}{\partial t} + \cancel{m^2c^2} - \left[\left(\frac{\partial S'}{\partial x}\right)^2 + \left(\frac{\partial S'}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right]$

$c \to +\infty$

$= \frac{\partial S'}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S'}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] = 0$

$\longrightarrow$ 底速下, Hamilton-Jacob方程.

角动量.

$\vec{M} = \sum \vec{r_i} \times \vec{p_i}$ 　空间各向同性

小转动下不变

无穷小转动 $\delta\Omega^{ik}$

$x'^i - x^i = x_k \delta\Omega^{ik}$ ①

径矢长度不变. $x^i x_i = x'^i x'_i$

$x'_i - x_i = x^k \delta\Omega_{ik}$ ②

$$x'^i x'_i - x'^i x_i - x^i x'_i + x^i x_i = 0$$

$$\Rightarrow x^i x^k \delta\Omega_{ik} = 0$$

$$x^k x^i \delta\Omega_{ki} = 0$$

$$x^i x^k (\delta\Omega_{ik} + \delta\Omega_{ki}) = 0$$

$\delta\Omega_{ik} \longrightarrow$ 二阶反对称张量

$$\delta S = -\sum m c u^i \delta x_i \Big|_a^b = -\sum p^i \delta x_i \Big|_a^b$$

$$\delta x_i = \delta\Omega_{ik} x^k$$

$$\delta S = -\delta\Omega_{ik} \sum p^i x^k \Big|_a^b$$

$$p^i x^k = \frac{1}{2}(p^i x^k - p^k x^i) + \frac{1}{2}(p^i x^k + p^k x^i)$$

$$\delta S = -\delta\Omega_{ik} \frac{1}{2} \sum (p^i x^k - p^k x^i) \Big|_a^b = 0$$

封闭系统

$$M^{ik} = \sum (p^i x^k - p^k x^i)$$

$$M^{01},\ M^{02},\ M^{03} \longrightarrow t\vec{p} - \frac{\mathcal{E}\vec{r}}{c^2}$$

$$M^{23} = M_x,\quad M^{13} = -M_y,\quad M^{12} = M_z$$

$$M^{ik} = \left(c\sum\left(t\vec{p} - \frac{\mathcal{E}\vec{r}}{c^2}\right),\ -\vec{M} \right)$$

$$\sum\left(t\vec{p} - \frac{\mathcal{E}\vec{r}}{c^2}\right) = \text{Const},\qquad \sum \mathcal{E} = \text{Const}$$

$$t\,\frac{c^2\sum_i \vec{p}_i}{\sum_i \mathcal{E}_i} - \frac{\sum_i \mathcal{E}_i \vec{r}_i}{\sum_i \mathcal{E}_i} = \text{Const}.$$

$$\vec{R} = \frac{\sum_i \mathcal{E}_i \vec{r}_i}{\sum_i \mathcal{E}_i} \qquad \frac{d\vec{R}}{dt} = \frac{c^2\sum_i \vec{p}}{\sum_i \mathcal{E}} = \vec{v}$$

$C \longrightarrow t\,\vec{v}$ $\qquad X_c \qquad\qquad v_c.$

三. 电磁场中的电荷

1. 场的四维势

小Tips: 相对论中无刚体

$\Downarrow$

"基本"粒子 ⟶ 无尺度的几何点

↳ $q_i, \dot{q}_i$, $i=1,2,3$

经典(非量子)

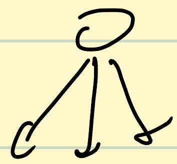
先作约定:

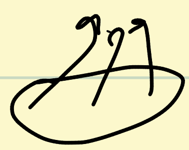
粒子与电磁场的相互作用的性质

↓ 电荷 e　　↓ 四维势 A_i (x_i 的函数)

$$-\frac{e}{c}\int_a^b A_i dx^i$$

类似公理





$$S=\int_a^b\left(-mc\,ds-\frac{e}{c}A_i dx^i\right)$$

$$A^i=(A^0,\vec{A})$$

↓ 标势 φ　　↓ 矢势 $\vec{A}$

$$A^i=(\varphi,\vec{A})\qquad A_i=(\varphi,-\vec{A})$$

$$S=\int_a^b\left(-mc\,ds-e\varphi\,dt+\frac{e}{c}\vec{A}\cdot d\vec{r}\right)$$

$\vec{v}=\frac{d\vec{r}}{dt}$,　$ds=c\,dt\sqrt{1-\frac{v^2}{c^2}}$

$$S=\int_{t_1}^{t_2}\left(-mc^2\sqrt{1-\frac{v^2}{c^2}}-e\varphi+\frac{e}{c}\vec{A}\cdot\vec{v}\right)dt$$

$$L=-mc^2\sqrt{1-\frac{v^2}{c^2}}+\frac{e}{c}\vec{A}\cdot\vec{v}-e\varphi$$ ⟶ 描述粒子与电磁场的相互作用

$$\vec{P}=\frac{\partial L}{\partial \vec{v}}=\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{e}{c}\vec{A}=\vec{p}+\frac{e}{c}\vec{A}$$

$$\mathcal{H}=\vec{v}\cdot\vec{P}-L=\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}+e\varphi$$

$$\left(\frac{\mathcal{H}-e\varphi}{c}\right)^2=m^2c^2+\left(\vec{P}-\frac{e}{c}\vec{A}\right)^2$$

$$\hookrightarrow \mathcal{H}=\sqrt{m^2c^4+c^2\left(\vec{P}-\frac{e}{c}\vec{A}\right)^2}+e\varphi.$$

$c\to+\infty$

$$L\to\frac{1}{2}mv^2+\frac{e}{c}\vec{A}\cdot\vec{v}-e\varphi$$

$$\vec{P}\to m\vec{v}+\frac{e}{c}\vec{A}$$

$$\mathcal{H}\to\frac{1}{2m}\left(\vec{P}-\frac{e}{c}\vec{A}\right)^2+e\varphi$$

Hamilton-Jacob方程.

$$\mathcal{H}=-\frac{\partial S}{\partial t},\quad \vec{P}=\frac{\partial S}{\partial \vec{r}}=\nabla S$$

$$\left(\frac{\partial S}{\partial t}+e\varphi\right)^2=m^2c^4+c^2\left(\nabla S-\frac{e}{c}\vec{A}\right)^2$$

2. 运动方程.

假定. $e \xrightarrow{\text{不影响}}$ 场 ⟶ 经典电动力学的适用范围

E-L方程

$$\frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \vec{v}}\right)}^{\vec{P}}=\frac{\partial L}{\partial \vec{r}}\longrightarrow \nabla L=\frac{e}{c}\nabla(\vec{A}\cdot\vec{v})-e\nabla\varphi$$

$$\vec\nabla(\vec A\cdot\vec v) = (\vec A\cdot\cancel{\vec\nabla})\vec v + \underline{(\vec v\cdot\vec\nabla)\vec A} + \underline{\vec A\times(\cancel{\vec\nabla\times\vec v})} + \underline{\vec v\times(\vec\nabla\times\vec A)}$$

$$\frac{\partial L}{\partial \vec r} = \frac{e}{c}(\vec v\cdot\vec\nabla)\vec A + \frac{e}{c}\vec v\times(\vec\nabla\times\vec A) - e\vec\nabla\varphi$$

$$\frac{d}{dt}\vec P = \frac{d}{dt}\left(\vec p + \frac{e}{c}\vec A\right) = \frac{\partial L}{\partial \vec r} \;//$$

$$d\vec A = \frac{\partial \vec A}{\partial t}dt + \frac{\partial \vec A}{\partial \vec r}d\vec r$$

$$\frac{d\vec A}{dt} = \frac{\partial \vec A}{\partial t} + \frac{\partial \vec A}{\partial \vec r}\cdot\frac{d\vec r}{dt} \quad \left(\frac{d\vec r}{dt}\to\vec v\right)$$

$$\frac{\partial \vec A}{\partial x}\frac{dx}{dt} + \frac{\partial \vec A}{\partial y}\frac{dy}{dt} + \frac{\partial \vec A}{\partial z}\frac{dz}{dt}$$

$$= \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\vec A$$

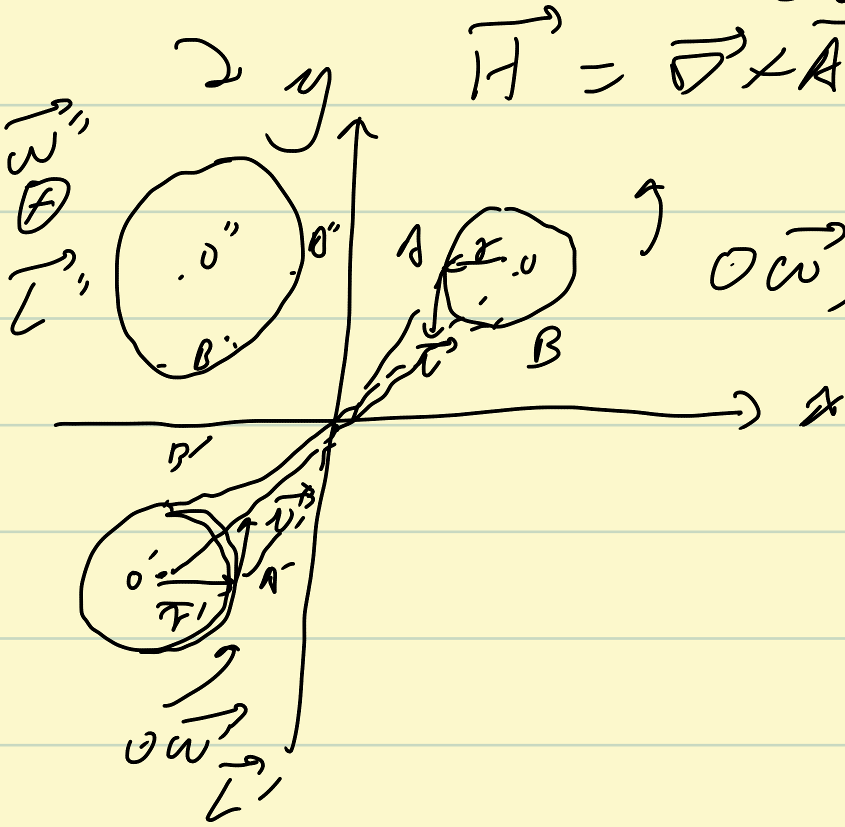
$$= (\vec v\cdot\vec\nabla)\vec A$$

$$\frac{d\vec p}{dt} + \frac{e}{c}\left[\frac{\partial \vec A}{\partial t} + (\vec v\cdot\vec\nabla)\vec A\right] = \frac{e}{c}(\vec v\cdot\vec\nabla)\vec A + \frac{e}{c}\vec v\times(\vec\nabla\times\vec A) - e\vec\nabla\varphi$$

$$\vec F = \frac{d\vec p}{dt} = \underbrace{-\frac{e}{c}\frac{\partial \vec A}{\partial t} - e\vec\nabla\varphi}_{\text{与}\vec v\text{无关}} + \underbrace{\frac{e}{c}\vec v\times(\vec\nabla\times\vec A)}_{\perp\text{与}\vec v\text{有关}}$$

定义 $\vec E = -\frac{1}{c}\frac{\partial \vec A}{\partial t} - \vec\nabla\varphi$ 极矢量

$\vec H = \vec\nabla\times\vec A$ 轴矢量(赝矢量)



⊙$\vec\omega$, $\vec L = m\,\vec r\times\vec v$
$\qquad\qquad\;\;\downarrow\;\;\downarrow$

$A^i = (\varphi, \vec A)$
$\downarrow$
极矢量

⊙z

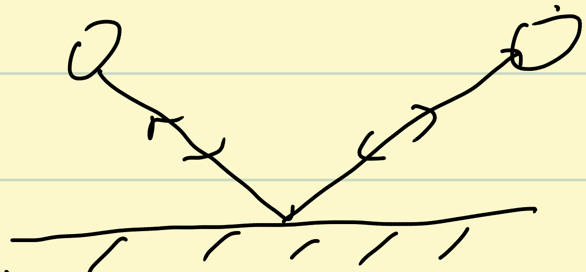
$$\frac{d\vec p}{dt} = \underbrace{e\vec E}_{\parallel \vec E} + \underbrace{\frac{e}{c}\vec v\times\vec H}_{\perp \vec H \quad \perp \vec v}$$

$$\mathcal{E}_{kin} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \vec p = \frac{m\vec v}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{d\mathcal{E}_{kin}}{dt} = \vec v\cdot\frac{d\vec p}{dt} = \vec v\cdot\left(e\vec E + \frac{e}{c}\vec v\times\vec H\right) = e\vec v\cdot\vec E$$

$$dW = e\vec E\cdot d\vec r$$

时间反演



$$\frac{d\vec p}{dt} = e\vec E + \frac{e}{c}\vec v\times\vec H \qquad t=-t \quad \vec E=\vec E, \quad \vec H=-\vec H$$

3. 规范不变性

$$\frac{d\vec p}{dt} = e\vec E + \frac{e}{c}\vec v\times\vec H \qquad (\vec E,\vec H)\leftarrow(\varphi,\vec A)$$

$$A_k' = A_k - \frac{\partial f}{\partial x^k}$$

$$\begin{cases} \varphi' = \varphi - \frac{1}{c}\frac{\partial f}{\partial t} \\ \vec A' = \vec A + \nabla f \end{cases} \longrightarrow \text{所有的方程在这个势变换下不变}$$

规范不变性

4. 电磁场张量

$$S = \int_a^b \left(-mc\,ds - \frac{e}{c}A_i dx^i\right)$$

$$\delta S = -\int_a^b \left(mc\frac{dx_i\, d\delta x^i}{ds} + \frac{e}{c}A_i d\delta x^i + \frac{e}{c}\delta A_i dx^i\right)$$

$$= -\int \Big[d(mc u_i) - mc\,du_i\delta x^i) + d\big[\tfrac{e}{c}A_i\delta x^i\big] - \frac{e}{c}dA_i\delta x^i + \frac{e}{c}\delta A_i dx^i$$

$$\delta S = -\left(mcu_i + \frac{e}{c}A_i\right)\delta x^i + \int\left(mc\,du_i\,\delta x^i + \frac{e}{c}dA_i\,\delta x^i - \frac{e}{c}\delta A_i\,dx^i\right)$$

$$\delta S = \int\left(mc\,du_i\,\delta x^i + \frac{e}{c}dA_i\,\delta x^i - \frac{e}{c}\delta A_i\,dx^i\right)$$

$$du_i = \frac{du_i}{ds}ds \qquad dA_i = \frac{\partial A_i}{\partial x^k}dx^k \qquad \delta A_i = \frac{\partial A_i}{\partial x^k}\delta x^k.$$

$$\delta S = \int\left[mc\frac{du_i}{ds}ds + \frac{e}{c}\frac{\partial A_i}{\partial x^k}dx^k\,\delta x^i - \frac{e}{c}\frac{\partial A_i}{\partial x^k}\delta x^k\,dx^i\right]$$

傀指标

$$\downarrow$$

$$\frac{e}{c}\frac{\partial A_k}{\partial x^i}\delta x^i\,dx^k$$

$$dx^k = u^k ds$$

$$\delta S = \int\underbrace{\left[mc\frac{du_i}{ds} - \frac{e}{c}\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right)u^k\right]}_{=0}\delta x^i\,ds = 0.$$

$$mc\frac{du_i}{ds} - \frac{e}{c}\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right)u^k = 0$$

引入电磁场张量 $F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$ (二阶反对称)

$$mc\frac{du_i}{ds} = \frac{e}{c}F_{ik}u^k$$

$$mc\frac{du^i}{ds} = \frac{e}{c}F^{ik}u_k \qquad \text{四维形式}$$

$$A_i = (\varphi, -\vec{A}).\qquad \vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{H} = \vec{\nabla}\times\vec{A} = \Big(\underbrace{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}}_{H_x},\ \underbrace{\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}}_{H_y},\ \underbrace{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}}_{H_z}\Big)$$

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

i ↓ 行, k ↓ 列

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}$$

$$F_{12} = -\Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big) = -H_z$$

$$F_{13} = -\Big(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\Big) = H_y$$

$$F_{23} = -\Big(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\Big) = -H_x$$

$$F_{ik} = (\vec{E}, \vec{H})$$

$$F^{ik} = (-\vec{E}, \vec{H})$$

$$mc\frac{du^i}{ds} = \frac{e}{c}F^{ik}u_k \qquad i = 0, 1, 2, 3$$

$i = 1, 2, 3$

$$\hookrightarrow \frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{H}.$$

$i = 0$

$$\hookrightarrow \frac{d\mathcal{E}_{kin}}{dt} = e\vec{E}\cdot\vec{v}$$

$$\delta S = -\Big(mcu_i + \frac{e}{c}A_i\Big)\delta x^i$$

$$P_i = -\frac{\partial S}{\partial x^i} = \underbrace{mcu_i}_{\hookrightarrow p_i} + \frac{e}{c}A_i = p_i + \frac{e}{c}A_i$$

$$P_i = \Big(\frac{\mathcal{E}_{kin} + e\varphi}{c},\ \vec{p} + \frac{e}{c}\vec{A}\Big)$$

$$\mathcal{E}_{总} = \mathcal{E}_{kin} + e\varphi.$$

5. 场的 Lorentz 变换

F^{ik}

$$A^{ik} \longrightarrow A'^{ik}$$
$$\parallel \qquad\qquad \parallel$$
$$A^i A^k \longrightarrow A'^i A'^k$$

$$A^i = (A^0, A^1, A^2, A^3)$$

$$a = \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} \qquad L = \begin{bmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$A'^i \longrightarrow A^i$$

$$a = L a'$$

$$A^{ik} = a \cdot a^T = \begin{bmatrix} A^{00} & A^{01} & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$A^{ik} = A^i A^k$$

$$A^{01} = A^0 \cdot A^1$$
$$A^{10} = A^1 A^0$$

$$a = L a'$$
$$a^T = (L a')^T = a'^T L^T$$
$$A = a a^T = L a' a'^T L^T = L A' L^T = L A' L$$

$$\begin{bmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} A'^{00} & A'^{01} & A'^{02} & A'^{03} \\ A'^{10} & A'^{11} & A'^{12} & A'^{13} \\ A'^{20} & A'^{21} & A'^{22} & A'^{23} \\ A'^{30} & A'^{31} & A'^{32} & A'^{33} \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma A'^{00} + \gamma\beta A'^{10} & \gamma A'^{01} + \gamma\beta A'^{11} & \gamma A'^{02} + \gamma\beta A'^{12} & \gamma A'^{03} + \gamma\beta A'^{13} \\ \gamma\beta A'^{00} + \gamma A'^{10} & \gamma\beta A'^{01} + \gamma A'^{11} & \gamma\beta A'^{02} + \gamma A'^{12} & \gamma\beta A'^{03} + \gamma A'^{13} \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$$

$$A^{10} = \gamma^2\beta A'^{00} + \gamma^2 A'^{10} + \gamma^2\beta^2 A'^{01} + \gamma^2\beta A'^{11}$$

$$A^{20} = \gamma A'^{20} + \gamma\beta A'^{21}$$

$$A^{30} = \gamma A'^{30} + \gamma\beta A'^{31}$$

$$A^{32} = A'^{32}$$

$$A^{13} = \gamma\beta A'^{03} + \gamma A'^{13}$$

$$A^{21} = \gamma\beta A'^{20} + \gamma A'^{21}$$

$$F^{ik} = \begin{bmatrix} & & & \\ E_x & & & H_y \\ E_y & H_z & & \\ E_z & & H_x & \end{bmatrix}$$

$$\begin{cases} E_x = E'_x \\ E_y = \gamma(E'_y + \beta H'_z) = \dfrac{E'_y + \frac{v}{c}H'_z}{\sqrt{1-v^2/c^2}} \\ E_z = \gamma(E'_z - \beta H'_y) = \dfrac{E'_z - \frac{v}{c}H'_y}{\sqrt{1-v^2/c^2}} \end{cases} \qquad \begin{cases} H_x = H'_x \\ H_y = \dfrac{H'_y - \frac{v}{c}E'_z}{\sqrt{1-v^2/c^2}} \\ H_z = \dfrac{H'_z + \frac{v}{c}E'_y}{\sqrt{1-v^2/c^2}} \end{cases}$$

6. 场的不变量

F^{ik} $\qquad$ $F^{ik}F_{ik} = \text{Const.}$

$$F^{ik} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{bmatrix} = A = \begin{bmatrix} 0 & -E \\ E^T & H \end{bmatrix}$$

$$F_{ik} = B = \begin{bmatrix} 0 & E \\ -E^T & H \end{bmatrix} \qquad A \cdot B^T = \begin{bmatrix} -E^2 & \\ & -E^TE + HH^T \end{bmatrix}$$

$$F^{ik}F_{ik} = -F^{ik}F_{ki} = -\mathrm{tr}(A\cdot B)$$

$$F^{ik}F_{ik} = \mathrm{tr}(A\cdot B^T)$$

$$F^{ik}F_{ik} = 2(H^2 - E^2) \longrightarrow H^2 - E^2 = \text{Const.}$$

$$e^{iklm} = \begin{cases} 1 & N(iklm) \to 偶 \\ -1 & 奇 \\ 0 & \end{cases}$$

② $e^{iklm} F_{ik} F_{lm} = Const.$

赝标量: $\vec{E} \cdot \vec{H} = Const$

K, $\vec{E} \cdot \vec{H} = 0 \Leftrightarrow \vec{E} \perp \vec{H} \longrightarrow K': \vec{E}' \perp \vec{H}'$

$K: |\vec{E}| = |\vec{H}| \longrightarrow K' \; |\vec{E}'| = |\vec{H}'|$.

$K_0 \quad \vec{E}_0 \cdot \vec{H}_0 \neq 0 \longrightarrow K', \; \vec{E}' // \vec{H}'$

$$\begin{cases} \vec{E}' \cdot \vec{H}' = |\vec{E}'||\vec{H}'| = \vec{E}_0 \cdot \vec{H}_0 \\ (\vec{E}')^2 - (\vec{H}')^2 = \vec{E}_0^2 - \vec{H}_0^2 \end{cases}$$

e.g. 求K'相对K的速度$\vec{V}$

四. 电磁场方程

$$\begin{cases}\vec{H} = \nabla \times \vec{A} \\ \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \varphi\end{cases}$$

$$\begin{cases}\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} & ① \\ \nabla \cdot \vec{H} = 0. & ②\end{cases}$$

第一对 Maxwell 方程

$\vec{H}$, $\vec{E}$, $\frac{\partial \vec{H}}{\partial t}$ $\frac{\partial \vec{E}}{\partial t}$

②:

$$\underbrace{\iiint_V \nabla \cdot \vec{H}}_{0} dV = \oiint_{\partial V} \vec{H} \cdot \vec{n}\, d\sigma = 0.$$

→ 磁场的高斯定理.

①.

$$\iint_D (\nabla \times \vec{E}) \cdot \vec{n}\, d\sigma = \underbrace{\oint_{\partial D} \vec{E} \cdot d\vec{l}}_{电动势} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_D \vec{H} \cdot \vec{n}\, d\sigma$$

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

$$\frac{\partial F_{ik}}{\partial x^l} = \frac{\partial^2 A_k}{\partial x^l \partial x^i} - \frac{\partial^2 A_i}{\partial x^l \partial x^k}$$

$$\frac{\partial F_{kl}}{\partial x^i} = \frac{\partial^2 A_l}{\partial x^i \partial x^k} - \frac{\partial^2 A_k}{\partial x^i \partial x^l}$$

$$\frac{\partial F_{li}}{\partial x^k} = \frac{\partial^2 A_i}{\partial x^k \partial x^l} - \frac{\partial^2 A_k}{\partial x^l \partial x^i}$$

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0$$

$i \quad k \quad l$

$$\left.\begin{matrix}0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 3\end{matrix}\right\} \rightarrow \nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = 0$$

$$1 \quad 2 \quad 3 \rightarrow \nabla \cdot \vec{H} = 0.$$

电磁场的作用量

$$S = S_m + S_{mf} + S_f \longrightarrow \text{无电荷时,场的作用量}$$

$S_m \to -\sum mc\int ds$

$S_{mf} \to -\sum \frac{e}{c}\int A_k dx^k$

实验原理: 叠加原理

↓

场方程是线性的

$$0 = \delta S = \delta \int \underbrace{\quad}_{\text{场的二次型}}$$

$\underline{F_{ik}F^{ik}}$

$$S_f = \underbrace{a}_{\text{常数}} \iint \underbrace{F_{ik}F^{ik}}_{2(H^2-E^2)} dV dt$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \varphi$$

高斯单位制: $a = -\frac{1}{16\pi}$

$$S_f = \frac{1}{8\pi}\iint (E^2 - H^2) dV dt \qquad d\Omega = c\,dt\,dx\,dy\,dz$$

$$= -\frac{1}{16\pi c}\int F_{ik}F^{ik} d\Omega$$

$$S = -\sum \int mc\,ds - \sum \int \frac{e}{c} A_k dx^k - \frac{1}{16\pi c}\int F_{ik}F^{ik} d\Omega$$

$\longrightarrow$ 定义 四维电流矢量

电荷密度 ρ. $\sum e_i = \int \rho \, dV$

$\rho = \sum_i e_i \delta(\vec{r} - \vec{r}_i)$

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

四维电流矢量

$$j^i = \rho \frac{dx^i}{dt} = (c\rho, \underbrace{\rho \vec{v}}_{\vec{j}})$$

$$\sum e_i = \int \rho \, dV = \frac{1}{c} \int j^0 dV = \frac{1}{c} \int j^i dS_i$$

$dS_i \to$ 与 x^i 垂直三维超平面

$dS_0 = dV$

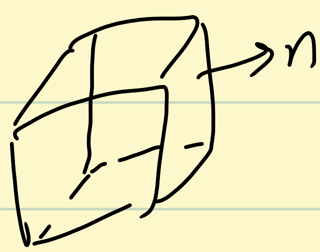
$dS_3 = c\, dt\, dx\, dy$

$$-\sum \int \frac{e}{c} A_k dx^k$$

$$= -\frac{1}{c} \int \rho \frac{dx^i}{dt} A_i \, dV dt$$

$$= -\frac{1}{c^2} \int \rho j^i A_i \, d\Omega$$

连续性方程



$$\sum e_i = \int \rho \, dV$$

$$\frac{\partial}{\partial t} \int \rho \, dV = -\oint_{\partial V} \vec{j} \cdot \vec{n} \, d\sigma$$

$$\frac{\partial}{\partial t}\int \rho dV = -\oint \vec{j}\cdot\vec{n}\, d\sigma = -\int \vec{\nabla}\cdot\vec{j}\, dV.$$

$$\int \left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j}\right) dV = 0$$

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = 0 \longrightarrow \text{连续性方程}$$

$$\rho = e\,\delta(\vec{r}-\vec{r}_0)$$

$$\vec{j} = \rho\vec{v} = e\vec{v}\,\delta(\vec{r}-\vec{r}_0) \qquad \vec{v} = \frac{\partial \vec{r}_0}{\partial t}.$$

$$\frac{\partial \rho}{\partial \vec{r}_0} = \lim_{\Delta r\to 0}\frac{\rho[\vec{r}-(\vec{r}_0+\Delta\vec{r})] - \rho(\vec{r}-\vec{r}_0)}{\Delta r}$$

$$= \lim_{\Delta r\to 0}\frac{\rho([\vec{r}+(-\Delta\vec{r})] - \vec{r}_0) - \rho(\vec{r}-\vec{r}_0)}{\Delta r}$$

$$= -\frac{\partial \rho}{\partial \vec{r}} = -\vec{\nabla}\rho$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\rho\cdot\vec{v} = -\vec{\nabla}\cdot(\rho\vec{v})$$

$$j^i = (c\rho, \vec{j})$$

$$\underline{\frac{\partial j^i}{\partial x^i} = 0} \longrightarrow \text{电荷守恒}.$$

$$\Updownarrow$$

$$\Sigma e_i = \frac{1}{c}\int j^i dS_i \qquad \oint_{\partial\Omega} j^i dS_i = 0$$

$$\parallel$$

$$\int_\Omega \frac{\partial j^i}{\partial x^i} d\Omega = 0$$

$$\parallel$$

$$0$$

规范不变性： $A_i - \frac{\partial f}{\partial x^i}$

$$S = -\sum\int mc\,ds - \frac{1}{c^2}\int A_i j^i d\Omega - \frac{1}{16\pi c}\int F_{ik}F^{ik} d\Omega.$$

$$A_i - \frac{\partial f}{\partial x^i}$$

$$\frac{1}{c^2}\int \frac{\partial f}{\partial x^i} j^i d\Omega$$

$$\parallel$$

$$\frac{1}{c^2}\int \frac{\partial (f j^i)}{\partial x^i} d\Omega = \frac{1}{c^2}\oint f j^i dS_i$$

第二对Maxwell方程

→ 运动方程，假定场已知，变分粒子的轨道.

→ 场方程，假定粒子轨道已知，变分 A_i（势）

$$\delta S = -\frac{1}{c}\int\left[\frac{1}{c}j^i\delta A_i + \frac{1}{16\pi}(\delta F_{ik}F^{ik} + F_{ik}\delta F^{ik})\right]d\Omega$$

$$\underbrace{\qquad}_{2F^{ik}\delta F_{ik}}$$

$$= -\frac{1}{c}\int\left[\frac{1}{c}j^i\delta A_i + \frac{1}{8\pi}F^{ik}\delta F_{ik}\right]d\Omega$$

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \qquad \delta F_{ik} = \frac{\partial}{\partial x^i}\delta A_k - \frac{\partial}{\partial x^k}\delta A_i$$

$$\delta S = -\frac{1}{c}\int\left\{\frac{1}{c}j^i\delta A_i + \underbrace{\frac{1}{8\pi}F^{ik}\frac{\partial}{\partial x^i}\delta A_k}_{-\frac{1}{8\pi}F^{ik}\frac{\partial}{\partial x^k}\delta A_i} - \frac{1}{8\pi}F^{ik}\frac{\partial}{\partial x^k}\delta A_i\right\}d\Omega$$

$$= -\frac{1}{c}\int\left[\frac{1}{c}j^i\delta A_i - \frac{1}{4\pi}F^{ik}\frac{\partial}{\partial x^k}\delta A_i\right]d\Omega.$$

$$\int F^{ik}\frac{\partial}{\partial x^k}\delta A_i d\Omega = \qquad -\int\frac{\partial F^{ik}}{\partial x^k}\delta A_i d\Omega$$

$$\delta S = -\frac{1}{c}\int \underbrace{\left[\frac{1}{c}j^i + \frac{1}{4\pi}\frac{\partial F^{ik}}{\partial x^k}\right]}\delta A_i d\Omega = 0.$$

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c}j^i$$

① $i=0$ $\quad \nabla\cdot\vec{E} = 4\pi\rho$ —— III

② $i=1,2,3$ $\quad \frac{1}{c}\frac{\partial\vec{E}}{\partial t} - \nabla\times\vec{H} = -\frac{4\pi}{c}\vec{j}$

$$\nabla\times\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j} \text{ —— IV.}$$

III: $\iiint_V \nabla\cdot\vec{E}\, dV = \underbrace{\oiint_{\partial V}\vec{E}\cdot\vec{n}\, d\sigma}_{} = 4\pi\int\rho\, dV.$

→ 电通量的高斯定理.

IV. $\iint_D (\nabla\times\vec{H})\cdot\vec{n}\, d\sigma = \oint_{\partial D}\vec{H}\cdot d\vec{l} = \frac{1}{c}\frac{\partial}{\partial t}\int\vec{E}\cdot\vec{n}\, d\sigma + \frac{4\pi}{c}\int\vec{j}\cdot\vec{n}\, d\sigma$

$\frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$ → 位移电流

$$= \frac{4\pi}{c}\int\left(\vec{j} + \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}\right)\cdot\vec{n}\, d\sigma.$$

→ 磁场沿闭合回路的环流

= $\frac{4\pi}{c}$ ·(电流 + 位移电流)

$$\nabla\cdot(\nabla\times\vec{H}) = \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\vec{E} + \frac{4\pi}{c}\nabla\cdot\vec{j}$$

$$= \frac{4\pi}{c}\left(\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t}\right)$$

$$0 = \frac{\partial^2 F^{ik}}{\partial x^i\partial x^k} = -\frac{4\pi}{c}\frac{\partial j^i}{\partial x^i}$$

总结：Maxwell方程组

3维形式 | 4维形式

$$\begin{cases} \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \\ \nabla\cdot\vec{H} = 0 \\ \nabla\times\vec{H} = \frac{4\pi}{c}\left(\vec{j} + \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}\right) \\ \nabla\cdot\vec{E} = 4\pi\rho \end{cases}$$

i,k,l 有0 ; i,k,l 1,2,3 $\Rightarrow$ $\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0$

$i=1,2,3$; $i=0$ $\Rightarrow$ $\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c}j^i$

Maxwell $\longrightarrow$ 四维张量方程的形式

电磁场本身的性质.

$$\vec{H}\cdot(\nabla\times\vec{E}) = -\frac{1}{c}\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = -\frac{1}{2c}\frac{\partial H^2}{\partial t}$$

$$\vec{E}\cdot(\nabla\times\vec{H}) = \frac{4\pi}{c}\vec{j}\cdot\vec{E} + \frac{1}{2c}\frac{\partial E^2}{\partial t}$$

$$\nabla\cdot(\vec{E}\times\vec{H}) = \vec{E}\cdot(\nabla\times\vec{H}) - \vec{H}\cdot(\nabla\times\vec{E}) = \frac{1}{2c}\frac{\partial(E^2+H^2)}{\partial t} + \frac{4\pi}{c}\vec{j}\cdot\vec{E}$$

$$\frac{\partial}{\partial t}\left(\frac{E^2+H^2}{8\pi}\right) = -\vec{j}\cdot\vec{E} - \nabla\cdot\Big(\underbrace{\frac{c}{4\pi}\vec{E}\times\vec{H}}_{\text{坡印亭矢量 }\vec{S}}\Big)$$

$$\frac{\partial}{\partial t}\int_V \frac{E^2+H^2}{8\pi}dV = -\int\vec{j}\cdot\vec{E}\,dV - \oint_{\partial V}\vec{S}\cdot\vec{n}\,d\sigma$$

$$-\sum_i\frac{d\mathcal{E}_{kin}}{dt} = -\sum_i e_i\vec{v}_i\cdot\vec{E}$$

$\vec{j} = \rho\vec{v}$

$\int\rho\vec{v}\cdot\vec{E}\,dV$

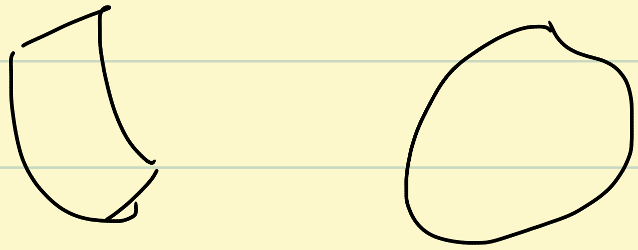
$\int\rho\,dV = \sum e_i$

$e\vec{v}\cdot\vec{E} = \frac{d}{dt}\mathcal{E}_{kin}$

$$\frac{d}{dt}\Big[\underbrace{\int\frac{E^2+H^2}{8\pi}dV}_{\text{电磁场的量}} + \sum_i\mathcal{E}_{kin}\Big] = 0$$

电磁场的量 $W = \frac{E^2+H^2}{8\pi}$ 能量密度

$$\frac{\partial}{\partial t}\left[\int \frac{E^2+H^2}{8\pi}\, dV + \sum_i \mathcal{E}_{kin}\right] = -\oint_{\partial V} \vec{S}\cdot\vec{n}\, d\sigma .$$



$\vec{S} \longrightarrow$ 能流密度

一般情况.

$$S = \underline{\int \Lambda\left(q, \frac{\partial q}{\partial x^i}\right)} dV\, dt = \frac{1}{c}\int \Lambda\, d\Omega$$

$$\hookrightarrow \Lambda = -\frac{1}{16\pi} F_{kl}F^{kl} \qquad q = A_i =$$

$$S = \int L\, dt \qquad L = \int \Lambda\, dV \qquad q_{,i} = \frac{\partial q}{\partial x^i}$$

$$\delta S = \frac{1}{c}\int \left(\frac{\partial \Lambda}{\partial q}\delta q + \frac{\partial \Lambda}{\partial q_{,i}}\delta q_{,i}\right) d\Omega$$

$$= \frac{1}{c}\int \left[\frac{\partial \Lambda}{\partial q}\delta q + \frac{\partial}{\partial x^i}\left(\frac{\partial \Lambda}{\partial q_{,i}}\delta q\right) - \delta q \frac{\partial}{\partial x^i}\frac{\partial \Lambda}{\partial q_{,i}}\right] d\Omega$$

$$\frac{\partial}{\partial x^i}\frac{\partial \Lambda}{\partial q_{,i}} - \frac{\partial \Lambda}{\partial q} = 0 \xrightarrow{E-L} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

$$\delta^k_i \frac{\partial \Lambda}{\partial x^k} = \frac{\partial \Lambda}{\partial x^i} = \frac{\partial \Lambda}{\partial q}\frac{\partial q}{\partial x^i} + \frac{\partial \Lambda}{\partial q_{,k}}\frac{\partial q_{,k}}{\partial x^i}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial \Lambda}{\partial q_{,k}}\right) q_{,i} + \frac{\partial \Lambda}{\partial q_{,k}}\frac{\partial q_{,k}}{\partial x^i}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial \Lambda}{\partial q_{,k}} q_{,i}\right)$$

$$\frac{\partial}{\partial x^k}\Big[\underbrace{q_{,i}\frac{\partial\Lambda}{\partial q_{,k}} - \delta_i^k\Lambda}_{=T_i^k}\Big] = 0$$

$$\frac{\partial A^k}{\partial x^k} = 0 \rightarrow \int A^k dS_k \text{ 守恒.}$$

$$P_i = \alpha\int T_i^k dS_k \qquad P^i = \alpha\int T^{ik} dS_k.$$

$i = 0$

$$P^0 = \alpha\int T^{0k} dS_k = \alpha\int T^{00} dV$$

$$T^{00} = \dot q\frac{\partial\Lambda}{\partial \dot q} - \Lambda \qquad P^i = \left(\frac{\mathcal{E}}{c}, \vec{P}\right).$$

令 $\alpha = \frac{1}{c}$

$$\underbrace{E}_{=\int T^{00}dV} = \dot q\frac{\partial L}{\partial \dot q} - L.$$

$$P^i = \frac{1}{c}\int T^{ik} dS_k$$

T^{ik} → 四维能量动量张量.

$$\underline{T^{ik} + \frac{\partial}{\partial x^l}\psi^{ikl}}$$

→ 四维角动量张量

$$M^{ik} = \int (x^i dP^k - x^k dP^i) = \frac{1}{c}\int (x^i T^{kl} - x^k T^{il}) dS_l$$

$$\frac{\partial}{\partial x^l}(x^i T^{kl} - x^k T^{il}) = 0$$

$$\frac{\partial x^i}{\partial x^l} = \delta_l^i, \qquad \frac{\partial T^{kl}}{\partial x^l} = 0$$

$$\rightarrow \delta_l^i T^{kl} - \delta_l^k T^{il} = 0 = T^{ki} - T^{ik}$$

$$P^i = \frac{1}{c}\int T^{i0}\,dV \qquad \underline{i=1,2,3}$$

$$\left(\frac{1}{c}T^{10}, \frac{1}{c}T^{20}, \frac{1}{c}T^{30}\right) \longrightarrow 动量密度$$

$T^{00} = W$，能量密度.

$$\frac{\partial T^{ik}}{\partial x^k} = 0 \longrightarrow \begin{cases} \frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0\alpha}}{\partial x^\alpha} = 0 \\ \frac{1}{c}\frac{\partial T^{\alpha 0}}{\partial t} + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0 \end{cases}$$

$\boxed{\alpha = 1,2,3}$

$\beta = 1,2,3$.

$$\frac{1}{c}\frac{\partial}{\partial t}\int T^{00}\,dV + \underline{\int \frac{\partial T^{0\alpha}}{\partial x^\alpha}\,dV} = 0$$

$$\parallel$$

$$\oint T^{0\alpha}\,df_\alpha. = (cT^{01}, cT^{02}, cT^{03})$$

$$\downarrow \vec{S}.$$

$$\frac{\partial}{\partial t}\int \frac{1}{c}T^{\alpha 0}\,dV = -\oint T^{\alpha\beta}\,df_\beta.$$

$T^{\alpha\beta} \longrightarrow$ 动量流密度张量.

$T^{\alpha\beta}$，单位时间内，⊥ x^β 单位面积的动量.

$\sigma_{\alpha\beta}$，应力张量

$$T^{ik} = \begin{bmatrix} W & \frac{S_x}{c} & \frac{S_y}{c} & \frac{S_z}{c} \\ \frac{S_x}{c} & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{S_y}{c} & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{S_z}{c} & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}$$

$$S = \int \Lambda\left(q, \frac{\partial q}{\partial x^i}\right) dV\,dt$$

$\longrightarrow$ 电磁场（无电荷）

$$\Lambda = -\frac{1}{16\pi} F_{kl} F^{kl}.$$

$$T_i^{\ k} = \frac{\partial q}{\partial x^i} \frac{\partial \Lambda}{\partial\left(\frac{\partial q}{\partial x^k}\right)} - \delta_i^k .$$

$$= \frac{\partial A_l}{\partial x^i} \boxed{\frac{\partial \Lambda}{\partial\left(\frac{\partial A_l}{\partial x^k}\right)}} - \delta_i^k \Lambda .$$

$$\delta \Lambda = -\frac{1}{8\pi} F^{kl} \delta F_{kl} = -\frac{1}{8\pi} F^{kl}\left(\delta \frac{\partial A_l}{\partial x^k} - \delta \frac{\partial A_k}{\partial x^l}\right).$$

$$= -\frac{1}{4\pi} F^{kl} \delta \frac{\partial A_l}{\partial x^k}$$

$$\frac{\partial \Lambda}{\partial\left(\frac{\partial A_l}{\partial x^k}\right)} = -\frac{1}{4\pi} F^{kl}$$

$$T_i^{\ k} = -\frac{1}{4\pi} \frac{\partial A_l}{\partial x^i} F^{kl} + \frac{1}{16\pi} \delta_i^k F_{lm} F^{lm}$$

$$T^{ik} = -\frac{1}{4\pi} \frac{\partial A^l}{\partial x_i} F^k_{\ l} + \frac{1}{16\pi} g^{ik} F_{lm} F^{lm} + \frac{\partial}{\partial x^l} \psi^{ikl}$$

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i = 0$$

构造 $\displaystyle \frac{\partial \psi^{ikl}}{\partial x^l} = \frac{1}{4\pi} \frac{\partial}{\partial x^l}\left(A^i F^{kl}\right) = \frac{1}{4\pi} \frac{\partial A^i}{\partial x^l} F^{kl}.$

$$\frac{\partial \psi^{ik}{}_l}{\partial x_l} = \frac{1}{4\pi} \frac{\partial A^i}{\partial x_l} F^k_{\ l}$$

$$T^{ik} = \frac{1}{4\pi}\left(-F^{il} F^k_{\ l} + \frac{1}{4} g^{ik} F_{lm} F^{lm}\right)$$

$$T_i^{\ i} = 0 \quad (i = 0, 1, 2, 3)$$

$$T^{00} = W = \frac{E^2+H^2}{8\pi}, \quad (cT^{01}, cT^{02}, cT^{03}) = \vec{S}$$

$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\left\{E_\alpha E_\beta + H_\alpha H_\beta - \frac{1}{2}\delta_{\alpha\beta}(E^2+H^2)\right\}$$

有带电粒子的情况下:

$$T_i^k = T^{(f)k}_{\ i} + T^{(p)k}_{\ i}$$

$m c u^\alpha \quad \underline{\alpha=1,2,3}$

$$\mu = \sum_i m_i \delta(\vec{r} - \vec{r}_i)$$

动量密度: $\mu c u^\alpha = \frac{T^{0\alpha}}{c}, \quad \alpha=1,2,3$

类比 $j^i = (c\rho, \vec{j}) = \rho \frac{dx^i}{dt}$

定义: 质量流矢量 $\mu \frac{dx^k}{dt} = (c\mu, \mu\vec{v})$.

$$\frac{\partial j^i}{\partial x^i} = 0 \iff \text{电荷守恒}$$

$$0 = \frac{\partial}{\partial x^k}\left(\mu \frac{dx^k}{dt}\right) \iff \text{质量守恒}$$

电荷在场中的运动方程.

$$m c \frac{du_i}{ds} = \frac{e}{c} F_{ik} u^k \qquad \frac{\mu}{m} = \frac{\rho}{e}.$$

$$\mu c \frac{du_i}{ds} = \frac{\rho}{c} F_{ik} u^k$$

$$\mu c \frac{du_i}{dt} = \frac{\rho}{c} F_{ik} \frac{dx^k}{ds}\frac{ds}{dt} = \frac{1}{c} F_{ik} j^k.$$

$$\frac{\partial}{\partial x^k}\left(T^{(p)k}_{\ i} + T^{(f)k}_{\ i}\right) = 0$$

$$\frac{\partial}{\partial x^k}\left(T^{(p)k}_{\ i}\right) = \frac{1}{4\pi}\Bigg(\frac{1}{2} F^{lm} \underbrace{\boxed{\frac{\partial F_{lm}}{\partial x^i}}}_{= -\frac{\partial F_{mi}}{\partial x^l} - \frac{\partial F_{il}}{\partial x^m}} - F^{kl}\frac{\partial F_{il}}{\partial x^k} - F_{il} \underbrace{\frac{\partial F^{kl}}{\partial x^k}}_{= \frac{4\pi}{c} j^l}\Bigg)$$

$$\frac{\partial}{\partial x^k}\left(T^{(p)k}_{\ i}\right) = -\frac{1}{4\pi}\Big[\underbrace{-\frac{1}{2}F^{lm}\frac{\partial F_{mi}}{\partial x^l} - \frac{1}{2}F^{lm}\frac{\partial F_{il}}{\partial x^m}}\ \cancel{- F^{kl}\frac{\partial F_{il}}{\partial x^k}} - \frac{4\pi}{c}F_{il}j^l\Big]$$

$$\downarrow$$

$$-F^{lm}\frac{\partial F_{il}}{\partial x^m} = \cancel{F^{ml}\frac{\partial F_{il}}{\partial x^m}}$$

$$\frac{\partial}{\partial x^k}\left(T^{(p)k}_{\ i}\right) = \frac{1}{c}F_{il}j^l$$

$$= \mu c\frac{du_i}{dt} = c\mu\frac{dx^k}{dt}\frac{\partial u_i}{\partial x^k} + c u_i\frac{\partial}{\partial x^k}\left(\mu\frac{dx^k}{dt}\right)$$

$$= \frac{\partial}{\partial x^k}\left(c u_i \mu\frac{dx^k}{dt}\right)$$

$$T^{(p)k}_{\ i} = \mu c\frac{dx_i}{ds}\frac{dx^k}{dt} = \mu c u_i u^k\frac{ds}{dt}.$$

位力定理:

$$T^{(f)i}_{\ i} = 0. \qquad T^i_i = T^{(p)i}_{\ i} = \mu c\,\underbrace{u^i u_i}_{1}\frac{ds}{dt}$$

$$= \mu c^2\sqrt{1-\frac{v^2}{c^2}}$$

$$T^i_i = \sum_a m_a c^2\sqrt{1-\frac{v_a^2}{c^2}}\,\delta(\vec{r}-\vec{r}_a) \geq 0$$

$$\frac{1}{c}\frac{\partial T^{\alpha 0}}{\partial t} + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0 \quad (\alpha,\beta = 1,2,3)$$

$$f(t) \qquad \overline{\frac{df}{dt}} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T\frac{df}{dt}\,dt = 0$$

$$x^\alpha\frac{\partial}{\partial x^\beta}\overline{T^\beta_\alpha} = 0.$$

$$0 = \int x^\alpha \frac{\partial}{\partial x^\beta} \overline{T_\alpha^\beta}\, dV$$

$$= \underbrace{\int \frac{\partial}{\partial x^\beta}\left(x^\alpha \overline{T_\alpha^\beta}\right) dV}_{\oint_{\partial V} x^\alpha \overline{T_\alpha^\beta}\, d\vec{f} = 0} - \int \underbrace{\frac{\partial x^\alpha}{\partial x^\beta}}_{\delta^\alpha_\beta} \overline{T_\alpha^\beta}\, dV$$

$$0 = \int \overline{T_\alpha^\alpha}\, dV.$$

$$\int \overline{T_i^i}\, dV = \int \underline{\left(\overline{T_0^0} + \overline{T_\alpha^\alpha}\right)} dV = \mathcal{E}$$

$$\mathcal{E} = \sum_i m_i c^2 \sqrt{1 - \frac{v_i^2}{c^2}}$$ 位力定理

$$\xrightarrow{c \to +\infty} \mathcal{E} - \sum_i m_i c^2 = -\sum_i \frac{m_i \overline{v_i^2}}{2}$$

专题：经典电动力学的适用范围

① 静电场 $\longrightarrow$ 无限大"自能"

② 运动电荷 $\longrightarrow$ 辐射阻尼

一. 静电场:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{E} = 0 \end{cases} \qquad \vec{E} = -\vec{\nabla}\varphi$$

$$\iiint \vec{\nabla}\cdot\vec{E}\,dV = \oiint_{\partial V} \vec{E}\cdot\vec{n}\,d\sigma = 4\pi\iiint \rho\,dV$$

e, R

库仑定律: $\vec{E} = \dfrac{e\vec{R}}{R^3}$ $\qquad \varphi = \sum_i \dfrac{e_i}{R_i} = \int \dfrac{\rho}{R}\,dV$

$$W = \frac{E^2}{8\pi}$$

$$U = \int W\,dV = \frac{1}{8\pi}\int E^2\,dV \qquad E = -\vec{\nabla}\varphi$$

$$= \frac{1}{8\pi}\int \vec{E}\cdot\vec{\nabla}\varphi\,dV$$

$$= \frac{1}{8\pi}\left[\int \varphi(\vec{\nabla}\cdot\vec{E})\,dV - \int \vec{\nabla}\cdot(\vec{E}\varphi)\,dV\right]$$

$$\int \vec{\nabla}\cdot(\vec{E}\varphi)\,dV = \oiint_{\partial V} \vec{E}\varphi\cdot\vec{n}\,d\sigma = 0$$

$$= \frac{1}{2}\int \rho\varphi\,dV$$

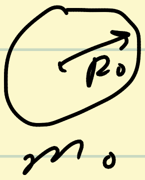
$$= \frac{1}{2}\sum_i e_i\varphi_i$$

只有一个电荷: $\dfrac{1}{2}e\varphi_0 \to \infty$ $\qquad \varphi = \dfrac{e}{R} \to +\infty$ $\quad (R \to 0)$

$$mc^2 = \frac{1}{2}e\varphi \to \infty$$

重正化: 引入非电磁源 "$-\infty$" 负质量

电荷→点.



$$\frac{e^2}{R_0} \sim m_0 c^2$$

→ 电子的"经典半径" $R_0 \sim \frac{e^2}{m_0 c^2}$

2. 辐射阻尼.

运动电荷的场.

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i$$

$$\frac{\partial^2 A^k}{\partial x_i \partial x^k} - \frac{\partial^2 A^i}{\partial x_k \partial x^k} = -\frac{4\pi}{c} j^i$$

$$\frac{\partial^2 A^i}{\partial x_k \partial x^k} = \frac{4\pi}{c} j^i \qquad \frac{\partial A^i}{\partial x^i} = 0$$

$$\downarrow$$

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{A} = 0$$

$i=0$ $\quad \Delta \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t} = -4\pi \rho$

$i=1,2,3$ $\quad \Delta \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t} = -\frac{4\pi}{c}\vec{j}$

→ 找特解. dV, $de(t)$. $\quad de \quad dV$

$$\rho = de(t)\,\delta(\vec{R})$$

$$\Delta \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t} = -4\pi\, de(t)\,\delta(\vec{R})$$

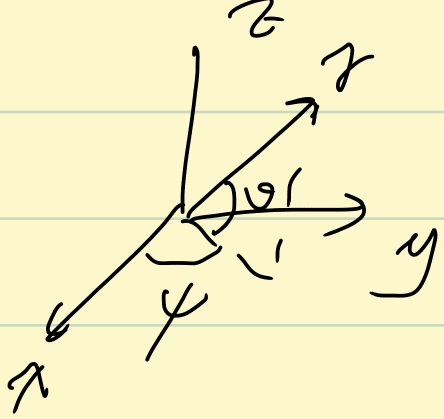
$\vec{R} \neq 0$, $\delta(\vec{R}) = 0$.

$$\Delta \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t} = 0$$

→ 球坐标. $\qquad h_i = \left\|\frac{\partial \vec{r}}{\partial q_i}\right\|$, Lamé 系数.

正交曲线坐标系 (q_1, q_2, q_3)

$$\Delta f = \sum_i \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial q_i}\left(\frac{h_1 h_2 h_3}{h_i^2}\frac{\partial f}{\partial q_i}\right)$$



$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases}$$

(r, θ, φ)

$h_1 = 1,\ h_2 = r,\ h_3 = r.$

$$\longrightarrow \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \varphi}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0$$

$$\chi(R,t) = \varphi R$$

$$\longrightarrow \frac{\partial^2}{\partial R^2}\chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = 0$$

$$\left(\frac{\partial}{\partial R} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial R} - \frac{1}{c}\frac{\partial}{\partial t}\right)\chi = 0$$

令 $\xi = t - \frac{R}{c}$, $\eta = t + \frac{R}{c}$

$$\begin{cases} \frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial R} = \frac{1}{c}\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) \end{cases}$$

$$\frac{\partial^2 \chi}{\partial \xi \partial \eta} = 0$$

$$\chi = f_1(\xi) + f_2(\eta) = f_1\left(t - \frac{R}{c}\right) + \underset{0}{\underset{\parallel}{f_2}}\left(t + \frac{R}{c}\right)$$

$$\varphi = \frac{\chi\left(t - \frac{R}{c}\right)}{R}.$$

$$\chi(t) = de(t)$$

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \underline{-4\pi\, de(t)\,\delta(\vec{R})}$$

$\vec{R} \to 0.$

$$\Delta\varphi = -4\pi\, de(t)\,\delta(\vec{R}) \qquad \varphi = \frac{de(t)}{R}$$

$$\varphi = \frac{de(t-\frac{R}{c})}{R} \qquad de = \rho\, dV.$$

$$\varphi(\vec{r}, t) = \int \frac{1}{R}\rho(\vec{r}\,', t-\frac{R}{c})\, dV' + \varphi_0$$

$$\vec{R} = \vec{r} - \vec{r}\,' \qquad dV' = dx'dy'dz'$$

$$\varphi = \int \frac{\rho_{t-\frac{R}{c}}}{R} dV + \varphi_0$$

$$\vec{A} = \frac{1}{c}\int \frac{\vec{j}_{t-\frac{R}{c}}}{R} dV + \vec{A}_0$$

推迟势

→ L 精确到 $(\frac{v}{c})$ 三阶及以上

$$L_a = -m_a c^2\sqrt{1-\frac{v_a^2}{c^2}} - e_a\varphi + \frac{e_a}{c}\vec{A}\cdot\vec{v}_a.$$

$(\frac{v}{c})$

$$\rho(t-\frac{R}{c}) = \rho - \frac{\partial}{\partial t}(\frac{R}{c}\rho) + \frac{1}{2}\frac{\partial^2}{\partial t^2}(\frac{R^2}{c^2}\rho) - \frac{1}{6}\frac{\partial^3}{\partial t^3}(\frac{R^3}{c^3}\rho)$$

$$\vec{j}(t-\frac{R}{c}) = \rho\vec{v} - \frac{\partial}{\partial t}(\frac{R}{c}\vec{j})$$

$$\varphi^{(3)} = -\frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\rho\, dV$$

$$\vec{A}^{(3)} = -\frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{j}\, dV.$$

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial f}{\partial t}, \quad \vec{A}' = \vec{A} + \vec{\nabla} f$$

$$f = -\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int R^2\rho\, dV. \quad \Big|\ \varphi' = 0$$

$$\vec{A}'^{(2)} = -\frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{j}\,dV - \frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\nabla\int R^2 \rho\, dV.$$

$$= \underbrace{-\frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{j}\,dV}_{-\frac{1}{c^2}\sum_i e_i \dot{\vec{v}_i}} \underbrace{- \frac{1}{3c^2}\frac{\partial^2}{\partial t^2}\int \vec{R}\rho\, dV}_{+\frac{1}{3c^2}\sum e_i \dot{\vec{v}_i}}$$

$\nabla R^2 = 2\vec{R}$.

$\dot{\vec{R}} = -\vec{V}$.

$\vec{R} = \vec{r} - \vec{r_i}$

$\vec{v} = \dot{\vec{r}}$

$$\vec{A}'^{(2)} = -\frac{2}{3c^2}\sum e_i \dot{\vec{v}_i}$$

$$\vec{d} = \sum_i e_i \vec{r_i}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A}'^{(2)}\right) = \frac{2}{3c^3}\dddot{\vec{d}}$$

$$\vec{f} = e\vec{E} = \frac{2e}{3c^3}\dddot{\vec{d}}$$

$$\sum \vec{f}\cdot\vec{v} = \frac{2}{3c^3}\dddot{\vec{d}}\sum e_i \vec{v_i}$$

$$= \frac{2}{3c^3}\dddot{\vec{d}}\cdot\dot{\vec{d}}$$

$$= \frac{2}{3c^3}\frac{d}{dt}\left(\dot{\vec{d}}\cdot\ddot{\vec{d}}\right) - \frac{2}{3c^3}\left(\ddot{\vec{d}}\right)^2$$

$$\overline{\sum \vec{f}\cdot\vec{v}} = -\frac{2}{3c^3}\overline{\left(\ddot{\vec{d}}\right)^2} \longrightarrow \text{对外辐射.}$$

$\vec{f}$: 辐射阻尼.

一个电荷,

$$m\dot{\vec{v}} = \frac{2e^2}{3c^3}\ddot{\vec{v}}$$

$$\dot{\vec{v}} \propto e^{\frac{3mc^3}{2e^2}}$$

$\longrightarrow$ 根源.

$L(q,\dot{q},t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

$\downarrow\ \dddot{q}$

$$m\dot{\vec{v}} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{H} + \frac{2e^2}{3c^3}\ddot{\vec{v}}$$

$$\boxed{\dot{\vec{v}} = \frac{e}{m}\vec{E}}$$

$$\ddot{\vec{v}} = \frac{e}{m}\dot{\vec{E}} + \frac{e}{mc}\dot{\vec{v}}\times\vec{H}$$

$$\frac{e}{mc}\vec{v}\times\vec{H}$$

$$\ddot{\vec{v}} = \frac{e}{m}\dot{\vec{E}} + \frac{e^2}{m^2c}\vec{E}\times\vec{H}$$

$$\vec{f} = \frac{2e^3}{3mc^3}\dot{\vec{E}} + \frac{2e^4}{3m^2c^4}\vec{E}\times\vec{H}$$

经典半径

$$\frac{2e^3}{3mc^3}\dot{\vec{E}} \sim \frac{e^3\omega E}{mc^3} \ll eE \rightarrow \frac{e^2\omega}{mc^3} \ll 1$$

$$\lambda \sim \frac{c}{\omega}$$

$$\lambda \gg \boxed{\frac{e^2}{mc^2}}$$

$$\frac{2e^4}{3m^2c^4}\vec{E}\times\vec{H} \sim \frac{e^4EH}{m^2c^4} \ll eE \rightarrow H \ll \frac{m^2c^4}{e^3}$$

分析力学

Faradag-Maxwell → 场是本质、电子“场的源头”

J.J. Thomson 1897

发现电子

Lorentz: 场、电子同描述

$$\text{介质}\begin{cases}\vec{e} = \vec{E}\\ \vec{h} = \vec{B}\end{cases}$$

Einstein 1905

狭义相对论.

量子力学

QFT

1915
广义相对论

$$T^{(p)k}_{\ i} = \mu c \frac{dx^i}{ds}\frac{dx^k}{dt} = \mu c u^i u^k \frac{ds}{dt}.$$

位力定理:

$$T^{(f)i}_{\ i} = 0. \qquad T^i_{\ i} = T^{(p)i}_{\ i} = \mu c \underbrace{u_i u^i}_{1} \frac{ds}{dt}$$

$$= \mu c^2 \sqrt{1 - \frac{v^2}{c^2}}$$

$$T^i_{\ i} = \sum_i m_i c^2 \sqrt{1 - \frac{v_i^2}{c^2}}\, \delta(\vec{r} - \vec{r}_i) \geq 0.$$

$$\frac{1}{c}\frac{\partial T^{\alpha 0}}{\partial t} + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0, \qquad \alpha, \beta = 1, 2, 3$$

$f(t)$ 有界

$$\overline{\frac{df}{dt}} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{df}{dt} dt = \frac{f(T) - f(0)}{T}$$
$$= 0$$

$$\frac{\partial}{\partial x^\beta} \overline{T^\beta_\alpha} = 0$$

$$0 = \int x^\alpha \frac{\partial}{\partial x^\beta}(\overline{T^\beta_\alpha}) dV = \int \frac{\partial}{\partial x^\beta}(x^\alpha \overline{T^\beta_\alpha}) dV - \int \frac{\partial x^\alpha}{\partial x^\beta} \overline{T^\beta_\alpha} dV$$

$$\| $$

$$\oint_{\partial V} x^\alpha \overline{T^\beta_\alpha} df = 0$$

$$0 = \int \underbrace{\boxed{\frac{\partial x^\alpha}{\partial x^\beta}}}_{\delta^\alpha_\beta} \overline{T^\beta_\alpha} dV = \int \delta^\alpha_\beta \overline{T^\beta_\alpha} = \int \overline{T^\alpha_\alpha} dV.$$

$$\int T_i^i \, dV = \int (T_0^0 + T_\alpha^\alpha) \, dV = \mathcal{E}.$$

$$\mathcal{E} = \sum_i m_i c^2 \sqrt{1 - \frac{v_i^2}{c^2}} \quad \text{位力定理.}$$

$$\xrightarrow{c \to \infty} \quad \mathcal{E} - \sum_i m_i c^2 = -\sum_i \frac{m_i v_i^2}{2}$$

Arendelle-ftl.