# Outline: Spherical Freezing Analogy for Cosmology

## Spherical freezing model

The planar freezing–lake model provides an exact analogue of a radiation–dominated Friedmann equation through the evolution of the ice thickness $s(t)$.[2] Here we show that an analogous result arises in a fully three–dimensional geometry by considering a spherically symmetric freezing problem.

**Setup.** Consider an ice sphere of radius $R(t)$ growing into an infinite bath of supercooled water. The liquid temperature far from the interface is

$$T(r \to \infty) = T_\infty < T_\phi,$$

where $T_\phi$ is the equilibrium freezing temperature. The ice–water interface is located at

$$r = R(t),$$

with

$$T(R(t), t) = T_\phi.$$

We assume:

1. spherical symmetry,
2. quasi–static heat conduction,
3. negligible temperature gradient inside the ice.

**Temperature field in the liquid.** In the freezing problem the temperature field is governed by the heat diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T,$$

where $\kappa$ is the thermal diffusivity. In typical solidification processes the thermal diffusion time scale is much shorter than the time scale over which the interface position $R(t)$ evolves. Consequently the temperature field relaxes rapidly to a quasi–steady profile compared with the slow motion of the freezing front. Under this quasi–static approximation the time derivative may be neglected, reducing the heat equation to Laplace's equation

$$\nabla^2 T = 0,$$

which determines the instantaneous temperature distribution in each phase. Considering spherical symmetry, in the liquid region $r > R(t)$ the temperature obeys Laplace's equation

$$\nabla^2 T = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0.$$

The spherically symmetric solution is

$$T_w(r) = A + \frac{B}{r}.$$

Applying the boundary conditions

$$T_w(R) = T_\phi, \qquad T_w(\infty) = T_\infty,$$

gives

$$T_w(r) = T_\infty + (T_\phi - T_\infty)\frac{R}{r}.$$

The radial gradient is therefore

$$\frac{dT_w}{dr} = -(T_\phi - T_\infty)\frac{R}{r^2}.$$

At the interface

$$\left.\frac{dT_w}{dr}\right|_{R^+} = -\frac{T_\phi - T_\infty}{R}.$$

**Stefan condition.** The latent heat released during solidification must be conducted away through the liquid. The Stefan condition therefore gives

$$\rho_i L \dot{R} = -\lambda_w \left.\frac{dT_w}{dr}\right|_{R^+},$$

where $\rho_i$ is the ice density, $L$ the latent heat of fusion, and $\lambda_w$ the thermal conductivity of water.

Substituting the interface gradient yields

$$\rho_i L \dot{R} = \lambda_w \frac{T_\phi - T_\infty}{R}.$$

Defining the constant

$$\alpha \equiv \frac{\lambda_w (T_\phi - T_\infty)}{\rho_i L},$$

we obtain the spherical Stefan growth law

$$\dot{R} = \frac{\alpha}{R}. \tag{1}$$

**Exact solution.** Multiplying Eq. (1) by $R$ gives

$$R\dot{R} = \alpha.$$

Hence

$$\frac{d}{dt}(R^2) = 2\alpha,$$

so that

$$R^2(t) = R_0^2 + 2\lambda(t - t_0). \tag{2}$$

At late times the growth law becomes

$$R(t) \propto t^{1/2}.$$

**Friedmann analogue.** Define the spherical Hubble parameter

$$H_R \equiv \frac{\dot{R}}{R}.$$

Using Eq. (1) we obtain

$$H_R = \frac{\alpha}{R^2}, \qquad H_R^2 = \alpha^2 R^{-4}. \tag{3}$$

This has exactly the form of a Friedmann equation for a spatially flat universe containing radiation,

$$H^2 \propto a^{-4}.$$

With the identification

$$a(t) \longleftrightarrow R(t),$$

the spherical freezing problem therefore reproduces the radiation–era expansion law

$$a(t) \propto t^{1/2}.$$

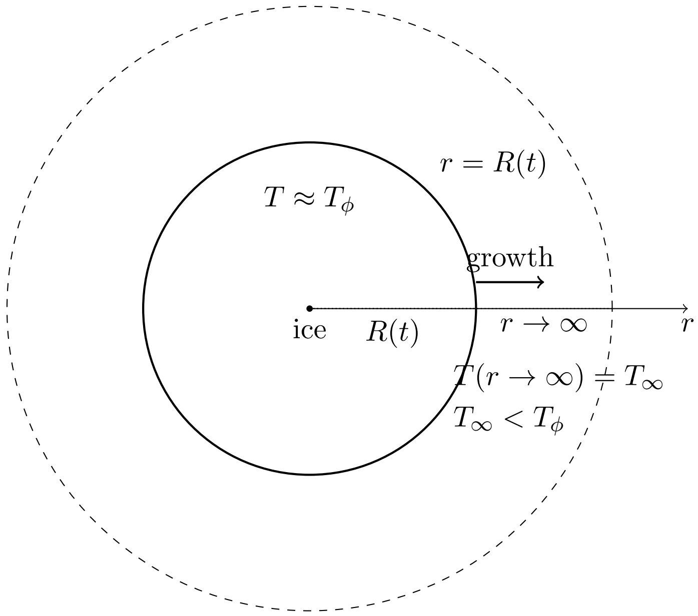


Figure 1: Spherical freezing model

**Effective energy density and equation of state.** Writing the evolution equation as

$$H_R^2 = \frac{8\pi G}{3}\rho_{\text{eff}}(R),$$

one finds

$$\rho_{\text{eff}}(R) = \frac{3\alpha^2}{8\pi G} R^{-4}.$$

Comparing with

$$\rho(a) \propto a^{-3(1+w)},$$

gives $w = \frac{1}{3}$.

Thus the spherical freezing model with an infinite thermal reservoir provides an *exact* analogue of a radiation–dominated Friedmann universe. This result parallels the planar freezing–lake model, but arises from the geometric spreading of radial heat flux rather than from the growth of a planar conductive layer.

## Spherical freezing with buoyancy-driven convection: an anisotropic Friedmann analogue

We extend the spherical freezing model by incorporating buoyancy-driven convection in the surrounding liquid, following the spirit of reduced transport models. Gravity introduces a preferred vertical direction, breaking exact spherical symmetry and leading naturally to an axisymmetric interface.

**Axisymmetric interface.** In the presence of gravity, the system remains axisymmetric about the vertical direction. The interface can therefore be written as a function $R(\theta, t)$ which is an axisymmetric function on the sphere, and thus admits an expansion in Legendre polynomials,

$$R(\theta, t) = \sum_{\ell=0}^{\infty} R_\ell(t)\, P_\ell(\cos\theta), \tag{4}$$

where $P_\ell$ are the eigenfunctions of the angular Laplacian,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_\ell}{d\theta}\right) = -\ell(\ell+1)P_\ell(\cos\theta),$$

and form a complete orthogonal basis for axisymmetric functions.

We separate the isotropic part by defining the mean radius

$$S(t) \equiv R_0(t), \tag{5}$$

and write the deformation as

$$\delta R(\theta, t) = R(\theta, t) - S(t) = \sum_{\ell\geq 1} R_\ell(t)\, P_\ell(\cos\theta). \tag{6}$$

The $\ell = 1$ mode corresponds to a translation of the center and does not represent a physical deformation, so it is discarded. For weak anisotropy, $|\delta R| \ll S$, higher-order modes are suppressed, and the leading nontrivial contribution is the quadrupole mode $\ell = 2$. Retaining only this term yields

$$R(\theta, t) = S(t)\big[1 + \varepsilon(t)P_2(\cos\theta)\big], \qquad |\varepsilon| \ll 1, \tag{7}$$

where we have defined $\varepsilon(t) = R_2(t)/S(t)$, and

$$P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$$

is the Legendre polynomial.

**Convective Stefan condition.** The local Stefan condition is modified by convective heat transport:

$$\rho_i L\, \partial_t R(\theta, t) = \frac{k_w \Delta T}{R(\theta, t)}\, Nu(\theta, t), \tag{8}$$

where $\Delta T = T_\phi - T_\infty$ and $Nu(\theta, t)$ is a local Nusselt factor.

We model the angular dependence as

$$Nu(\theta, t) = \bar{N}u(S)\big[1 + \xi(S)P_2(\cos\theta)\big], \tag{9}$$

with $\bar{N}u(S)$ the mean convective enhancement and $\xi(S)$ encoding anisotropy induced by buoyancy.

Define

$$\Lambda(S) \equiv \frac{k_w \Delta T}{\rho_i L}\, \bar{N}u(S). \tag{10}$$

**Evolution equations.** Expanding to first order in $\varepsilon$, the Stefan condition yields

$$\dot{S} = \frac{\Lambda(S)}{S}, \tag{11}$$

and

$$\dot{\varepsilon} = \frac{\Lambda(S)}{S^2}\,(\xi(S) - 2\varepsilon). \tag{12}$$

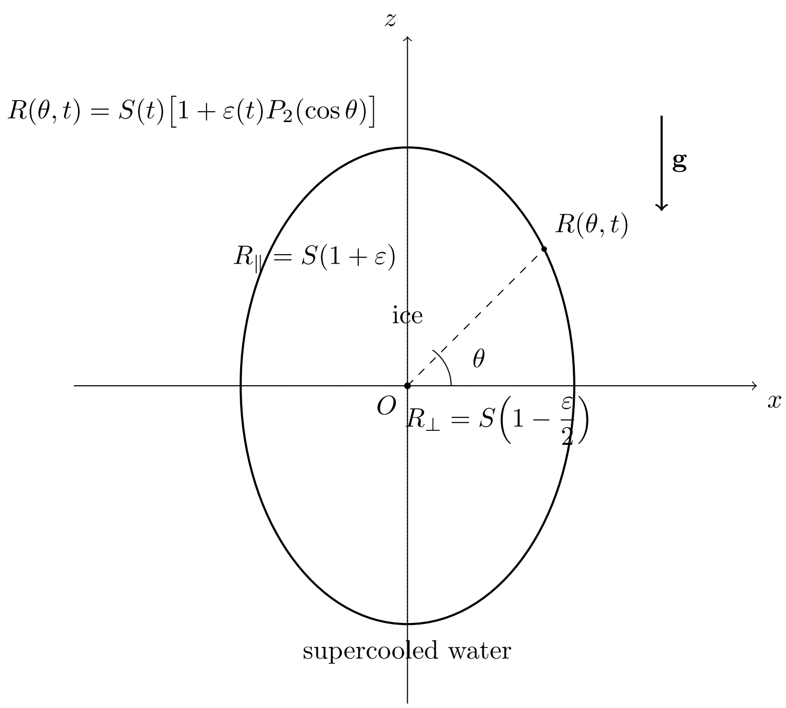


Figure 2: axisymmetric deformationfrom buoyancy-driven convection

**Mean Friedmann analogue.** Assuming a scaling law for convection,

$$\bar{N}u(S) \propto Ra(S)^{\beta}, \qquad Ra(S) \propto S^3, \tag{13}$$

one obtains

$$\Lambda(S) = \Lambda_0 S^{3\beta}. \tag{14}$$

The mean-radius evolution becomes

$$\dot{S} = \Lambda_0 S^{3\beta-1}, \qquad H \equiv \frac{\dot{S}}{S} = \Lambda_0 S^{3\beta-2}, \tag{15}$$

so that

$$H^2 \propto S^{6\beta-4}. \tag{16}$$

Identifying $S(t)$ with a cosmological scale factor, this corresponds to an effective equation of state

$$w_{\text{eff}} = \frac{1}{3} - 2\beta, \tag{17}$$

which reduces to radiation ($w = 1/3$) in the purely conductive limit $\beta = 0$.

**Anisotropic expansion.** Define directional scale factors

$$a_{\parallel} = S(1+\varepsilon), \qquad a_{\perp} = S\left(1 - \frac{\varepsilon}{2}\right), \tag{18}$$

so that $a_{\parallel} a_{\perp}^2 \approx S^3$.

The directional Hubble rates are

$$H_{\parallel} \approx H + \dot{\varepsilon}, \qquad H_{\perp} \approx H - \frac{1}{2}\dot{\varepsilon}. \tag{19}$$

The shear scalar behaves as

$$\sigma^2 \propto (H_{\parallel} - H_{\perp})^2 \propto \dot{\varepsilon}^2, \tag{20}$$

giving

$$\sigma^2 \propto S^{6\beta-4}(\xi - 2\varepsilon)^2. \tag{21}$$

**Modified Friedmann equation.** The parameter $\xi$ encodes anisotropic heat transport due to buoyancy. As the system evolves, mixing and geometric dilution reduce angular variations in heat flux, so that $\xi \to 0$ in the absence of sustained directional forcing. In this limit, the anisotropy equation then reduces to

$$\frac{d\varepsilon}{dS} = -\frac{2\varepsilon}{S}, \tag{22}$$

which integrates to

$$\varepsilon = \varepsilon_0 S^{-2}. \tag{23}$$

Using

$$\dot{S} = \Lambda_0 S^{3\beta-1}, \qquad H \equiv \frac{\dot{S}}{S} = \Lambda_0 S^{3\beta-2}, \tag{24}$$

together with

$$\sigma^2 = \frac{3}{4}\dot{\varepsilon}^2, \tag{25}$$

one finds

$$\dot{\varepsilon} = -2\varepsilon_0 \Lambda_0 S^{3\beta-4}, \tag{26}$$

and therefore

$$\sigma^2 = 3\Lambda_0^2 \varepsilon_0^2 \, S^{6\beta-8}. \tag{27}$$

The generalized Friedmann equation then takes the conservative form

$$3H^2 = 8\pi G\, \rho_{\mathrm{eff}}(S) + \sigma^2, \tag{28}$$

with

$$\frac{8\pi G}{3}\rho_{\mathrm{eff}}(S) = \Lambda_0^2 S^{6\beta-4}. \tag{29}$$

Hence

$$H^2 = \Lambda_0^2 S^{6\beta-4} + \Lambda_0^2 \varepsilon_0^2 S^{6\beta-8}. \tag{30}$$

Normalizing at the present epoch $S_0 = 1$, with

$$H_0^2 = \Lambda_0^2(1+\varepsilon_0^2), \tag{31}$$

and defining

$$\Omega_{c0} = \frac{1}{1+\varepsilon_0^2}, \qquad \Omega_{\sigma 0} = \frac{\varepsilon_0^2}{1+\varepsilon_0^2}, \tag{32}$$

one obtains

$$\frac{H^2}{H_0^2} = \Omega_{c0}\, S^{6\beta-4} + \Omega_{\sigma 0}\, S^{6\beta-8}. \tag{33}$$

This provides the minimal anisotropic extension of the spherical freezing analogue, with the second term playing the role of a shear contribution. In particular, for $\beta = 1/3$ the shear scales as $S^{-6}$, in direct analogy with Bianchi-I cosmology[1].

**Next step: Experiment/Simulation**

## References


[1] Ö. Akarsu, J. D. Barrow, and N. M. Uzun. Screening anisotropy via energy-momentum squared gravity: ΛCDM model with hidden anisotropy. *Phys. Rev. D*, 102(12):124059, 2020.

[2] L. F. Niehof, A. Venkatasubramanian, F. Toschi, and S. Liberati. Freezing lakes as analogue models of CDM cosmology and beyond. 2 2026.